DIFFERENTIAL CALCULUS

Theory, Techniques, and Engineering Applications

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Dedicated to Son and Daughter

Kosgei Arafat and Kosgei Adalyne

Preface

This book grew out of lecture notes developed over many years of teaching engineering mathematics at undergraduate level. Its central aim is to present differential calculus not as an abstract collection of rules and techniques, but as a living, coherent language — the language through which engineers describe, predict, and control the behaviour of physical systems.

The treatment is rigorous without being austere. Every formula is derived from first principles, because understanding the origin of a result is the surest way to apply it correctly in an unfamiliar setting. At the same time, every abstraction is anchored immediately to an engineering context — a turbine-blade temperature distribution, a vibrating mechanical system, a charging electrical circuit — so that the mathematics never becomes divorced from the physical reality it describes.

Scope and Organisation

The book is divided into eight chapters, each self-contained yet building on the preceding material:

Chapter 1 Foundations of Functions and Limits *establishes the real number system, the function concept, limit theory, and continuity — the rigorous underпinning for everything that follows.*

Chapter 2 The Concept of Differentiation *introduces the derivative as an instantaneous rate of change, develops it from the limit definition, and surveys the four standard engineering notations.*

Chapter 3 Fundamental Rules of Differentiation *derives and applies the Power, Product, Quotient, and Chain Rules, together with implicit, logarithmic, and parametric differentiation techniques.*

Chapter 4 Advanced Differentiation Techniques *extends the toolkit to trigonometric, exponential, logarithmic, inverse trigonometric, and hyperbolic functions, with full first-principles derivations.*

Chapter 5 Higher-Order Derivatives *covers nth derivatives, the Leibniz product rule, concavity, beam deflection theory, and vibration analysis.*

Chapter 6 Multivariable Differentiation *develops partial derivatives, the gradient, directional derivatives, the multivariable chain rule, and the vector operators of field theory.*

Chapter 7 Applications of Differentiation *applies calculus to optimisation, curve sketching, related rates, error propagation, and the Newton-Raphson method.*

Chapter 8 Engineering Modelling and Integrated Problems *presents complete derivations and case studies drawn from kinematics, thermal systems, fluid dynamics, electrical circuits, and structural mechanics.*

Prerequisites

Readers are assumed to have completed a standard secondary-level mathematics course covering algebra, trigonometry, and elementary functions. No prior knowledge of calculus is assumed; the book is self-contained from first principles.

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How to Use This Book

This book is designed equally for classroom instruction and independent study. The following notes will help readers navigate the material efficiently.

For Students

• Read each derivation actively: try to reproduce the key steps before reading on. A result understood through its derivation is far more durable than one merely memorised.

• Work every Worked Example before consulting the solution. A few minutes of genuine effort — even unsuccessful — is worth more than passive reading.

• Exercises are graded by difficulty. Those marked ★ are routine; ★★ require combination of rules; ★★★ are engineering-level challenge problems.

• Use the Master Derivative Reference Table in Appendix A to check results — but not as a substitute for understanding the derivations.

For Instructors

The eight chapters map naturally to a one-semester undergraduate course (approximately 48 contact hours). Chapters 1–3 form the core; Chapters 4–6 extend the theory; Chapters 7–8 provide the engineering integration. Chapters may be reordered or selectively omitted depending on the curriculum's focus.

Notation

Both the Leibniz notation (dy/dx, $\partial f/\partial x$) and the Lagrange notation ($f'(x)$, $f''(x)$) are used throughout and are treated as interchangeable. The choice in any given passage is guided by clarity and convention in the relevant engineering discipline. A complete notation guide is provided in Appendix B.

Worked Examples

Each Worked Example follows a Problem–Solution structure. The problem statement gives all data; the solution shows every algebraic step. Engineering problems include

dimensional analysis, unit conversions, and physical interpretation of results — elements that are considered integral to the solution, not optional extras.

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CHAPTER 1

Foundations of Functions and Limits

Real Numbers • Functions • Limits • Continuity • Engineering Applications

Mathematics is the language in which the laws of nature are written. Before an engineer can model the stress in a bridge cable, describe the decay of a radioactive isotope, or simulate the flow of current through a circuit, they must be fluent in this language. Chapter 1 builds that fluency from the ground up, beginning with the real number system — the arena in which all physical quantities live.

1.1 Real Numbers and the Real Line

1.1.1 Classification of Real Numbers

The real numbers, denoted $\mathbb{R}$, constitute the complete ordered field that underpins all of calculus. They are classified hierarchically:

- Natural Numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$: the counting numbers.
- Integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$: positive and negative whole numbers and zero.
- Rational Numbers $\mathbb{Q} = \{p/q : p, q \text{ integers}, q \neq 0\}$: numbers expressible as a fraction. Examples: $3/4$, $-7/2$, $0.333\ldots = 1/3$.
- Irrational Numbers: real numbers that cannot be written as p/q. Examples: $\sqrt{2}$, π, e, $\sqrt{5}$.
- Real Numbers $\mathbb{R} = \mathbb{Q} \cup \{\text{irrationals}\}$: the complete number line.

Completeness Axiom: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum) in $\mathbb{R}$. This property — absent from $\mathbb{Q}$ — guarantees that all limits exist and is the logical foundation of calculus.

1.1.2 Absolute Value, Intervals, and Engineering Tolerance

The absolute value $|x|$ measures the distance from x to the origin. It is defined piecewise: $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$. Key properties include the Triangle Inequality:

$|a+b| \le |a|+|b|$, and the distance interpretation: $|x - a|$ is the distance between x and a on the number line.

Absolute Value Inequalities: $|x - a| < \delta$ means $a - \delta < x < a + \delta$, i.e. x lies in the open interval $(a-\delta, a+\delta)$. This is the key to the ε-δ definition of a limit.

Intervals are connected subsets of $\mathbb{R}$. An open interval (a, b) excludes its endpoints; a closed interval [a, b] includes them. Half-open intervals include exactly one endpoint. Engineering tolerances are naturally expressed as absolute value inequalities.

Worked Example 1.1.1 — Solve $|3x - 7| < 4$. Solution: $-4 < 3x - 7 < 4 \Rightarrow 3 < 3x < 11 \Rightarrow 1 < x < 11/3$. Answer: $x \in (1, 11/3)$.

Worked Example 1.1.2 — A precision gear must satisfy $|d - 50| \le 0.02$ mm. This gives $49.98 \le d \le 50.02$ mm, a closed interval of acceptable diameters.

1.1.3 Inequalities, Sign Charts, and the Number Line

Solving algebraic inequalities is a fundamental skill that underpins the determination of domains, ranges, and regions of validity throughout calculus. The two principal methods are (1) algebraic manipulation combined with a sign chart, and (2) graphical analysis. The sign chart (or sign diagram) is the more systematic approach and generalises immediately to rational and polynomial inequalities of any degree.

Procedure — Sign Chart Method: (1) Rearrange the inequality so that one side is zero. (2) Factor the non-zero side completely. (3) Identify all real zeros and vertical asymptotes (the critical numbers). (4) These critical numbers partition the real line into open intervals. (5) Choose a test point in each interval and determine the sign of the expression. (6) Read off the solution from the sign pattern, being careful to include or exclude endpoints according to whether the inequality is strict or non-strict.

Worked Example 1.1.3 — Solve the inequality $(x-1)/(x+2) \ge 0$. Critical numbers: $x = 1$ (numerator zero) and $x = -2$ (denominator zero; excluded from solution). Sign chart on the three intervals: for $x < -2$, both factors are negative, so the fraction is positive (+). For $-2 < x < 1$, the ratio is negative (−). For $x > 1$, both factors are positive (+). The solution is $x \in (-\infty, -2) \cup [1, +\infty)$.

The sign chart method extends directly to polynomial inequalities: factor completely, locate all real roots with their multiplicities, and note that the sign changes at a root of odd multiplicity but does not change at a root of even multiplicity.

1.2 Functions and Mathematical Models

1.2.1 Definition and Notation

A function $f : A \to B$ is a rule that assigns to each element x in the domain A exactly one element $f(x)$ in the codomain B. The variable x is the independent variable (input); $y = f(x)$ is the dependent variable (output). The range (image) of f is the set of all values actually attained: $\text{Range}(f) = \{f(x) : x \in A\}$.

Vertical Line Test: A curve in the xy-plane represents a function of x if and only if every vertical line $x = c$ intersects the curve in at most one point.

1.2.2 Domain, Range, and Graphs

When no domain is specified, we use the natural domain: the largest subset of $\mathbb{R}$ for which $f(x)$ is real. Three main restrictions apply: (1) denominators cannot be zero; (2) even-index radicands must be non-negative; (3) logarithm arguments must be strictly positive.

Worked Example 1.2.1 — Find the domain and range of $g(t) = \sqrt{(16 - t^2)}$. Requirement: $16 - t^2 \geq 0 \Rightarrow t^2 \leq 16 \Rightarrow |t| \leq 4$. Domain = $[-4, 4]$. At $t = 0$, $g = 4$ (max); at $t = \pm 4$, $g = 0$ (min). Range = $[0, 4]$. This is the upper semicircle of radius 4.

1.2.3 Graph Transformations

Starting from a base function $y = f(x)$: replacing x by $(x - h)$ shifts the graph right h units; multiplying by a scales it vertically; negating gives a reflection over the x-axis; adding k shifts up.

Worked Example 1.2.2 — Sketch $h(x) = -2(x-3)^2 + 5$, starting from $y = x^2$. Steps: (1) shift right 3; (2) stretch by 2; (3) reflect; (4) shift up 5. Result: downward parabola with vertex $(3, 5)$, x-intercepts at $3 \pm \sqrt{(5/2)}$.

1.2.4 Standard Functions in Engineering

Engineering problems rely on a core set of function families. Each has characteristic behaviour that must be understood both analytically and graphically.

- Polynomials $P(x) = a_n x^n + \ldots + a_0$: continuous everywhere on $\mathbb{R}$. Used for stress–strain relations, trajectory equations, Taylor approximations.
- Rational functions $R(x) = P(x)/Q(x)$: continuous where $Q(x) \neq 0$. Arise in control system transfer functions and impedance models.

• Trigonometric: sin x, cos x and their relatives. Period 2π. Fundamental to AC circuit analysis, vibration, and wave superposition.
• Exponential $f(x) = e^x$ and logarithm $f(x) = \ln x$: inverses of each other. Model decay, growth, charging, and discharging in virtually every branch of engineering.
• Hyperbolic: $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$. The catenary (hanging cable) shape is $y = a\cosh(x/a)$.
• Piecewise and step functions: the Heaviside step $H(t)$ models instantaneous switching; ramp functions model linearly increasing inputs.

1.2.5 Composite Functions

The composite function $(f \circ g)(x) = f(g(x))$ feeds the output of g as the input to f. The domain of $f \circ g$ is $\{x \in \text{Dom}(g) : g(x) \in \text{Dom}(f)\}$. Composition is NOT commutative in general: $f \circ g \neq g \circ f$.

Worked Example 1.2.3 — Let $f(x) = \sqrt{x}$, $g(x) = 3x^2 - 12$. Then $(f \circ g)(x) = \sqrt{(3x^2 - 12)}$. Domain: $3x^2 - 12 \geq 0 \Rightarrow x^2 \geq 4 \Rightarrow |x| \geq 2$. $\text{Dom}(f \circ g) = (-\infty, -2] \cup [2, \infty)$. Compare: $(g \circ f)(x) = 3x - 12$, domain $[0, \infty)$. Completely different.

1.2.6 Inverse Functions

A function f is one-to-one (injective) if distinct inputs produce distinct outputs, equivalently every horizontal line hits the graph at most once. On this condition, f has an inverse function f^{-1} such that $f^{-1}(f(x)) = x$ for all x in the domain. Graphically, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

Worked Example 1.2.4 — Find the inverse of $f(x) = (2x + 1)/(x - 3)$. Set $y = (2x+1)/(x-3)$, solve for x: $y(x-3) = 2x+1 \Rightarrow yx - 3y = 2x+1 \Rightarrow x(y-2) = 3y+1 \Rightarrow x = (3y+1)/(y-2)$. Swapping variables: $f^{-1}(x) = (3x+1)/(x-2)$, $x \neq 2$.

Worked Example 1.2.5 (Beam Model) — A simply supported beam of span $L = 6$ m with UDL $w = 5$ kN/m has bending moment $M(x) = (wx/2)(L-x) = 15x - (5/2)x^2$ kN·m. Maximum $M = wL^2/8 = 22.5$ kN·m at $x = L/2 = 3$ m (midspan). Domain: $[0, 6]$. Range: $[0, 22.5]$.

1.2.7 Even and Odd Functions

A function f is even if $f(-x) = f(x)$ for all x in its domain, meaning its graph is symmetric about the y-axis. A function f is odd if $f(-x) = -f(x)$ for all x in its domain, meaning its

graph has 180° rotational symmetry about the origin. Most functions are neither even nor odd.

Standard examples: cos x and x^n (n even) are even; sin x, tan x, and x^n (n odd) are odd. Any function on a symmetric domain decomposes uniquely into an even part and an odd part: $f(x) = [f(x)+f(-x)]/2 + [f(x)-f(-x)]/2$. This decomposition underlies Fourier series.

Worked Example 1.2.6 (Classification) — (a) $f(x) = x^3 + 2x$ is odd: $f(-x) = -x^3 - 2x = -f(x)$. (b) $g(x) = x^4 - 3x^2 + 1$ is even: $g(-x) = g(x)$. (c) $h(x) = x^2 + x$ is neither. Engineering consequence: the integral of an odd function over any symmetric interval $[-a, a]$ is exactly zero.

1.3 Limits

1.3.1 Intuitive Concept

We write $\lim_{x\to a} f(x) = L$ and say "the limit of f(x) as x approaches a is L" if f(x) can be made arbitrarily close to L by taking x sufficiently close — but not equal — to a. The limit describes approach behaviour; the function value at the point itself is irrelevant to whether the limit exists.

A foundational example: $f(x) = (x^2 - 4)/(x - 2)$ is undefined at $x = 2$, but factoring gives $f(x) = x + 2$ for $x \neq 2$, so $\lim_{x\to 2} f(x) = 4$. The hole in the graph does not prevent the limit from existing.

1.3.2 The Fundamental Trigonometric Limit

$\lim_{x\to 0} \sin(x)/x = 1$ (x in radians). This is proved geometrically by trapping sin(x)/x between cos(x) and 1 using area inequalities on the unit circle, then applying the Squeeze Theorem. It justifies the engineering small-angle approximation $\sin(\theta) \approx \theta$.

1.3.3 The Formal ε-δ Definition

Definition 1.3.1 — $\lim_{x\to a} f(x) = L$ means: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

Interpretation: ε is any prescribed tolerance band around L on the output axis. δ is the corresponding closeness requirement on the input axis. The condition $0 < |x - a|$ explicitly excludes $x = a$ — limits are about approach, not arrival.

Worked Example 1.3.1 (Formal Proof) — Prove $\lim_{x\to 3}(4x - 7) = 5$. Analysis: $|(4x-7)-5| = |4x-12| = 4|x-3|$. We want $4|x-3| < \varepsilon$, so $|x-3| < \varepsilon/4$. Proof: Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x-3| < \delta$ then $|(4x-7)-5| = 4|x-3| < 4\cdot(\varepsilon/4) = \varepsilon$. QED.

1.3.4 Limit Laws and Computation Techniques

Once limits of simple functions are established by ε-δ, the following laws allow us to build up limits algebraically. If $\lim f(x) = L$ and $\lim g(x) = M$ (both as $x \to a$), then:

- Sum/Difference: $\lim [f(x) \pm g(x)] = L \pm M$
- Product: $\lim [f(x)\, g(x)] = LM$
- Quotient: $\lim [f(x)/g(x)] = L/M$, provided $M \neq 0$
- Power/Root: $\lim [f(x)]^n = L^n$; $\lim [f(x)]^{(1/n)} = L^{(1/n)}$ (appropriate conditions)
- Direct Substitution: if f is a polynomial or rational function and a is in its domain, $\lim_{x\to a} f(x) = f(a)$

When direct substitution gives $0/0$ or ∞/∞ (indeterminate forms), we must manipulate the expression first. The two main algebraic techniques are factoring & cancellation, and rationalisation (multiply by the conjugate).

1.3.5 Special Limits

Several special limits appear repeatedly in engineering mathematics and deserve memorisation:

- $\lim_{x\to 0} \sin(x)/x = 1$
- $\lim_{x\to 0} (1 - \cos x)/x^2 = 1/2$
- $\lim_{x\to 0} (e^x - 1)/x = 1$
- $\lim_{x\to 0} \ln(1+x)/x = 1$
- $\lim_{x\to \infty} (1 + k/x)^x = e^k$
- $\lim_{x\to \infty} x^p/e^x = 0$ (exponential growth overtakes any polynomial)

1.3.6 The Squeeze Theorem

Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ near a (excluding possibly a itself), and $\lim f(x) = \lim h(x) = L$, then $\lim g(x) = L$.

Worked Example 1.3.2 — $\lim_{x\to 0} x^2 \sin(1/x)$. Since $-1 \le \sin(1/x) \le 1$, we have $-x^2 \le x^2 \sin(1/x) \le x^2$. Both bounds $\to 0$, so by the Squeeze Theorem the limit = 0.

1.3.7 One-Sided Limits

The left-hand limit $\lim_{x \to a^-} f(x)$ considers only x approaching from the left ($x < a$); the right-hand limit $\lim_{x \to a^+}$ considers x approaching from the right. The two-sided limit exists if and only if both one-sided limits exist and are equal.

1.3.8 Limits at Infinity and Horizontal Asymptotes

$\lim_{x \to +\infty} f(x) = L$ if f(x) approaches L as x grows without bound. The horizontal line $y = L$ is a horizontal asymptote. For rational functions, divide numerator and denominator by the highest power of x in the denominator.

Worked Example 1.3.3 — $\lim_{x \to \infty} (4x^3 - 2x + 7)/(5x^3 + x^2 - 3)$. Divide by x^3: $\to (4 - 0 + 0)/(5 + 0 - 0) = 4/5$. General rule: if degrees are equal, limit = ratio of leading coefficients.

1.3.9 Non-Existence of Limits: Oscillatory Case

The function $f(x) = \sin(1/x)$ illustrates why a limit can fail to exist through oscillation. As $x \to 0$, the argument $1/x \to \pm\infty$, so the function oscillates between -1 and $+1$ infinitely often. Two sequences both approaching 0 produce function values that converge to different numbers, proving the limit does not exist.

1.3.10 Limits of Trigonometric Functions

Since sin x and cos x are continuous everywhere on the real line, direct substitution applies at all finite points where the function is defined. A general strategy for $\lim_{x \to 0} \sin(kx)/(mx)$ is to write it as $(k/m) \times \sin(kx)/(kx) \to k/m$.

Worked Example 1.3.4 — (a) $\lim_{x \to 0} \sin(3x)/(5x) = 3/5$. (b) $\lim_{x \to \pi/2} \cos(x)/(x - \pi/2)$: substitute $t = x - \pi/2$, giving $\lim_{t \to 0} -\sin(t)/t = -1$. (c) $\lim_{x \to 0} \tan(2x)/x = 2$. These patterns recur throughout vibration analysis and signal processing.

1.4 Continuity

1.4.1 Definition

Definition 1.4.1 — f is continuous at a if all three conditions hold: (1) f(a) is defined; (2) $\lim_{x\to a} f(x)$ exists; (3) $\lim_{x\to a} f(x) = f(a)$. If any condition fails, f has a discontinuity at a. f is continuous on an interval if it is continuous at every point in that interval.

Continuity can be visualised as "drawing the graph without lifting the pen." All polynomials, sin x, cos x, e^x, and ln x are continuous on their natural domains. Sums, products, quotients (where denominator is non-zero), and composites of continuous functions are continuous.

1.4.2 Types of Discontinuities

- Removable: the limit exists but f(a) is missing or wrong. Fixed by redefining f(a) = lim f(x).
- Jump: both one-sided limits exist but are unequal. Models instantaneous switching (Heaviside function). Cannot be fixed by redefining f at a single point.
- Infinite: at least one one-sided limit is ±∞. Creates a vertical asymptote. Models concentrated forces, electric fields near charges.
- Oscillatory (Essential): the limit fails to exist due to infinitely rapid oscillation, as in sin(1/x) at x = 0.

1.4.3 The Intermediate Value Theorem

Theorem 1.4.3 (IVT) — If f is continuous on [a, b] and N is strictly between f(a) and f(b), then there exists at least one $c \in (a, b)$ with f(c) = N. Corollary: if f(a) and f(b) have opposite signs, f has at least one root in (a, b).

The IVT powers the bisection method, the oldest and most reliable numerical root-finding algorithm, which is the foundation of many engineering simulation packages.

Worked Example 1.4.1 — Show $2x^3 - 5x + 1 = 0$ has a root in [0, 1]. f(0) = 1 > 0, f(1) = −2 < 0. By IVT, root exists. Bisection: f(0.5) = −1.25 < 0 ⇒ root in [0, 0.5]; f(0.25) = −0.22 < 0 ⇒ root in [0, 0.25]; f(0.125) = +0.38 > 0 ⇒ root in [0.125, 0.25]. Continuing gives root ≈ 0.205.

1.4.4 Uniform Continuity and the Extreme Value Theorem

Uniform Continuity: f is uniformly continuous on I if a single δ works for ALL pairs of points, not just locally.

Heine-Cantor Theorem: Every function continuous on a closed bounded interval [a, b] is uniformly continuous there. This underpins the convergence guarantees of finite element methods.

Extreme Value Theorem (EVT): If f is continuous on [a, b], then f attains both its global maximum and global minimum at some points in [a, b]. The EVT guarantees that maximum stress, peak temperature, and maximum deflection over a closed structural domain are genuinely attained — making design-against-maximum a well-posed engineering problem.

1.4.5 Continuity of Composite Functions and Trigonometric Functions

Theorem 1.4.5 (Continuity of Composite Functions) — If g is continuous at a and f is continuous at g(a), then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a. In limit notation: $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Since sin x and cos x are continuous everywhere, any composition such as $\sin(x^2 + 1)$, e^x, or $\sqrt{(\cos x + 2)}$ is automatically continuous wherever it is defined.

Worked Example 1.4.2 — Determine all discontinuities of $f(x) = \sqrt{(\sin x - 1)}$. The outer function $\sqrt{u}$ requires $u \geq 0$, so the domain is where $\sin x \geq 1$. Since $-1 \leq \sin x \leq 1$, this holds only when $\sin x = 1$, i.e., at $x = \pi/2 + 2n\pi$ for integer n. Within its domain f is continuous by the composition theorem.

1.5 Engineering Interpretation of Limits

1.5.1 Transient and Steady-State Behaviour

In every engineering system — electrical, mechanical, thermal — the response to an applied input has two parts: a transient component (which decays to zero as $t \to \infty$) and a steady-state component (which persists). The steady-state value is mathematically defined as the limit as $t \to \infty$.

Example: RC circuit charge Q(t) = CV0(1 − e^(−t/RC)). Steady state: $\lim_{t \to \infty} Q(t) = CV_0$. The time constant τ = RC marks the 63.2% charging point.

1.5.2 Small-Angle Approximations from Limits

All small-angle approximations used in engineering dynamics, optics, and mechanics are justified rigorously by limits. As $\theta \to 0$ (radians): $\sin(\theta) \approx \theta$ (from $\lim \sin(t)/t = 1$); $\cos\theta \approx 1 - \theta^2/2$ (from $\lim(1-\cos t)/t^2 = 1/2$); $\tan(\theta) \approx \theta$. These approximations convert nonlinear differential equations (e.g., the pendulum equation) into solvable linear ones.

1.5.3 Linearisation of Nonlinear Engineering Models

Linearisation replaces a nonlinear function $f(x)$ by its tangent line at an operating point x_0: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. This linear approximation is valid for small perturbations, and its error is a limit: $\lim_{x\to x_0} [f(x) - L(x)]/(x - x_0) = 0$.

Worked Example 1.5.1 — Linearise $f(V) = \sqrt{V}$ at operating point $V_0 = 9$ V. $f(9) = 3$, $f'(9) = 1/(2\sqrt{9}) = 1/6$. Linear model: $L(V) = 3 + (1/6)(V - 9)$. At $V = 10$: exact $= \sqrt{10} = 3.162$, linear $= 3.167$, error $= 0.14\%$ — excellent accuracy for a small perturbation of 1 V around 9 V.

1.5.4 Beam Bending and Piecewise Engineering Models

Real engineering systems often require piecewise function descriptions — different loading conditions in different regions, or elastic vs. plastic material behaviour. Limits and continuity determine whether the model is physically consistent at the boundaries between regions.

Example: Simply supported beam (length 6 m, UDL 5 kN/m) with bending moment $M(x) = (5x/2)(6-x)$. Maximum moment of 22.5 kN·m occurs at midspan $x = 3$ m.

1.5.5 Limits in Control and Signal Processing

In the frequency-domain analysis of linear control systems, the transfer function $G(j\Omega)$ describes the system's response at each frequency Ω. Two key limits are: the DC gain $\lim_{\chi\to 0} |G(j\Omega)|$ (steady-state response to a constant input), and the high-frequency behaviour $\lim_{\chi\to\infty} |G(j\Omega)|$ (noise rejection). For a first-order low-pass filter $G(j\Omega) = 1/(1+j\Omega\tau)$: DC gain = 1, high-frequency limit = 0 — the filter passes DC and attenuates fast oscillations.

In signal reconstruction, the sinc function $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ requires the limit $\lim_{x\to 0} \mathrm{sinc}(x) = 1$ to be defined and continuous at the origin. This continuity is essential to the Nyquist–Shannon sampling theorem.

Chapter 1 Summary

Key Concepts and Formulas

- Real Numbers: $\mathbb{R}$ is a complete ordered field. Completeness guarantees suprema and infima. Absolute value $|x-a|$ measures distance; $|x-a| < \delta$ gives the interval $(a-\delta, a+\delta)$.
- Functions: $f: A \to B$ assigns exactly one output to each input. Natural domain avoids zero denominators, even-root negatives, and zero log arguments. Composition and inversion are the fundamental function operations.
- Limits: $\lim_{x \to a} f(x) = L$ via ε-δ. Limit laws reduce complex limits to simple ones. Factoring, rationalising, and special limits resolve $0/0$ forms. One-sided limits must agree for the two-sided limit to exist.
- Special limits: $\lim \sin(x)/x = 1$, $\lim (1-\cos x)/x^2 = 1/2$, $\lim (e^x-1)/x = 1$ (all as $x \to 0$).
- Continuity: requires $f(a)$ defined + limit exists + limit = $f(a)$. IVT: continuous f on $[a,b]$ takes every value between $f(a)$ and $f(b)$. EVT: continuous f on $[a,b]$ attains its max and min.
- Engineering: limits describe steady-state, justify small-angle approximations, define derivatives and integrals, and guarantee existence of optimal design values.

Looking Ahead

In Chapter 2 we build directly on the limit concept to define the derivative: $f'(a) = \lim_{h \to 0} [f(a+h) - f(a)]/h$. This instantaneous rate of change is the central tool of differential calculus, and every technique in all subsequent chapters traces directly back to the foundations established here.

CHAPTER 2

The Concept of Differentiation

Rate of Change • First Principles • Differentiability • Engineering Notation

Differentiation is the branch of calculus concerned with finding the **instantaneous rate of change** of a function with respect to one of its variables. Together with integration, it forms the two fundamental pillars of calculus and underpins virtually every quantitative description of change in engineering and the physical sciences. Whether an engineer is computing the bending-moment distribution along a loaded beam, tracking the acceleration of a high-speed mechanism, modelling the transient temperature in a quenched casting, or analysing the voltage waveform in a switching circuit, differentiation provides the precise mathematical language in which such problems are formulated, solved, and interpreted.

The present chapter develops the concept of differentiation carefully and completely, starting from the physical notion of a rate of change and ending with a survey of the notational systems used across engineering disciplines. The approach is deliberately rigorous: every central formula is derived from first principles rather than simply stated, so that the student builds genuine conceptual understanding rather than a collection of rules to be applied without insight. At the same time, each abstract idea is anchored to a concrete engineering context — a turbine-blade temperature profile, a piston trajectory, a cooling curve — so that the mathematics is never divorced from the physical reality it describes.

By the end of this chapter the reader will be able to: define the derivative as a limit and compute it from first principles; derive the derivatives of the elementary functions — power, root, trigonometric, exponential, and logarithmic; distinguish rigorously between differentiability and continuity; find the equations of tangent and normal lines and interpret them geometrically and mechanically; interpret the first and second derivatives as velocity, acceleration, and thermodynamic rate; and read and write fluidly in the four standard engineering notations — Leibniz, Newton, Lagrange, and Euler.

2.1 Rate of Change in Physical Systems

Every physical quantity of engineering interest changes. Displacement changes with time, temperature changes with position, pressure changes with volume, current changes with voltage. The **rate of change** — how fast a quantity changes relative to another — is frequently more critical than the quantity itself. A turbine inlet temperature of 1400 °C may be perfectly acceptable in steady operation, but a temperature that is still climbing at 50 °C per second is dangerous: the blade material may not yet have reached its stress-failure regime, but it is approaching it rapidly. A pressure vessel at 150 bar is designed for that load, but if the pressure is rising at 20 bar per second a relief valve must respond within one second or failure will occur. In both cases the **rate**, not the level, drives the engineering decision.

2.1.1 The Average Rate of Change

Let $y = f(x)$ be a function defined on an interval. Suppose x changes from x_1 to x_2, producing a change in y from $f(x_1)$ to $f(x_2)$. We define:

$\Delta x = x_2 - x_1$ *(increment in the independent variable)*

$\Delta y = f(x_2) - f(x_1)$ *(corresponding increment in the dependent variable)*

Definition 2.1 — Average Rate of Change

The **average rate of change** of f with respect to x over $[x_1, x_2]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{[f(x^2) - f(x^1)]}{x^2 - x^1}$$

Writing $x_2 = x_1 + \Delta x$, equivalently:

$$\frac{\Delta y}{\Delta x} = \frac{[f(x^1 + \Delta x) - f(x^1)]}{\Delta x}$$

Geometrically, this ratio is the **slope of the secant line** joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f.

The average rate of change is the workhorse of numerical engineering: whenever two readings are taken from a data logger and a trend is needed, this formula is applied. Its limitation is equally obvious — it tells us the **mean** behaviour over an interval, not what is happening at a specific instant. A car's average speed over a 10-minute journey is not the same as its speed at the moment it passes a speed camera.

2.1.2 The Instantaneous Rate of Change

To obtain the rate of change at a **single point** $x = a$, we shrink the interval to zero by letting $\Delta x \to 0$. The secant line rotates progressively until, in the limit, it becomes the **tangent line** at (a, f(a)), and the ratio $\Delta y/\Delta x$ approaches the slope of that tangent. This limiting value is the **instantaneous rate of change** of f at $x = a$, and it is the derivative:

$$f'(a) = lim[\Delta x \to 0] \left(\frac{[f(a + \Delta x) - f(a)]}{\Delta x} \right)$$

The process of computing this limit is called **differentiation**, and its result is the **derivative** of f at a. Three questions immediately arise: does the limit always exist? Is the result unique? How do we compute it in practice? The rest of this chapter addresses all three.

2.1.3 Physical Units of a Rate of Change

A rate of change inherits its units from the quantities involved:

$$[\frac{dy}{dx}] = \frac{[units\ of\ y]}{[units\ of\ x]}$$

This dimensional rule extends automatically to higher derivatives: $[d^n y/dx^n] = [y]/[x]^n$. The rule is a powerful engineering check: any formula involving a derivative must have consistent dimensions, and an incorrect exponent or missing factor immediately reveals itself through dimensional analysis.

- f(t) = displacement (m), t = time (s) ⇒ f′(t) = velocity (m s^{-1}), f″(t) = acceleration (m s^{-2})
- T(x) = temperature (°C), x = distance (m) ⇒ dT/dx = temperature gradient (°C m^{-1})
- p(V) = pressure (Pa), V = volume (m^3) ⇒ dp/dV has units Pa m^{-3} (gas spring stiffness)
- q(t) = charge (C), t = time (s) ⇒ dq/dt = current (A)

Worked Example 2.1.1 — Average and Instantaneous Rate: Turbine Blade Temperature

Problem: The temperature distribution (in °C) along the chord of a gas-turbine aerofoil blade is measured from the leading edge at x = 0 to the trailing edge at x = 60 mm and fitted by the polynomial:

$$T(x) = 920 + 4.2x - 0.05x^2, 0 \leq x \leq 60\ mm$$

Worked Example 2.1.1 — Average and Instantaneous Rate: Turbine Blade Temperature

(a) Find the average temperature gradient from x = 10 mm to x = 30 mm. (b) Use the limit definition to find the instantaneous gradient at x = 20 mm. Express both answers in °C mm⁻¹.

Solution (a) — Average rate:

$$T(10) = 920 + 4.2(10) - 0.05(10)^2 = 920 + 42 - 5 = 957\,°C$$

$$T(30) = 920 + 4.2(30) - 0.05(30)^2 = 920 + 126 - 45 = 1001\,°C$$

$$\Delta T/\Delta x = (1001 - 957) / (30 - 10) = 44 / 20 = 2.20\,°C\,mm^{-1}$$

The average temperature gradient between 10 mm and 30 mm is **2.20 °C mm⁻¹**.

Solution (b) — Instantaneous rate at x = 20 mm:

$$T(20 + h) = 920 + 4.2(20 + h) - 0.05(20 + h)^2$$

$$= 920 + 84 + 4.2h - 0.05(400 + 40h + h^2)$$

$$= 920 + 84 + 4.2h - 20 - 2h - 0.05h^2$$

$$= 984 + 2.2h - 0.05h^2$$

$$T(20 + h) - T(20) = (984 + 2.2h - 0.05h^2) - 984 = 2.2h - 0.05h^2$$

$$[T(20 + h) - T(20)]/h = 2.2 - 0.05h$$

$$T'(20) = lim[h \to 0]\,(2.2 - 0.05h) = 2.20\,°C\,mm^{-1}$$

In this case the average and instantaneous rates coincide at the midpoint because T(x) is quadratic: the average rate over a symmetric interval equals the derivative at its midpoint (a consequence of the Mean Value Theorem, Chapter 4). Converting: 2.20 °C mm⁻¹ = 2200 °C m⁻¹. This is a large thermal gradient and indicates significant thermal stress along the blade chord.

Worked Example 2.1.2 — Instantaneous Drainage Rate from a Reservoir

Problem: A stormwater retention basin drains through an outlet sluice. The volume of water remaining (m³) at time t minutes is:

$$V(t) = 5000(1 - t/100)^2, 0 \le t \le 100\,min$$

Find: (a) the average drain rate over the first 25 minutes; (b) the instantaneous drain rate at t = 25 min; (c) the instantaneous drain rate when the basin is exactly half-empty.

(a) Average rate:

Worked Example 2.1.2 — Instantaneous Drainage Rate from a Reservoir

$$V(0) = 5000(1)^2 = 5000\,m^3$$

$$V(25) = 5000(1 - 0.25)^2 = 5000(0.5625) = 2812.5\,m^3$$

$$\frac{\Delta V}{\Delta t} = (2812.5 - 5000)/25 = -187.5/25 = -112.5\,m^3\,min^{-1}\,(i.e., 1.875\,m^3\,s^{-1})$$

(b) Instantaneous rate at t = 25:

$$V(25 + h) = 5000(1 - (25 + h)/100)^2 = 5000((75 - h)/100)^2$$

$$V(25 + h) - V(25) = 5000[(75 - h)^2 - 75^2]/10000 = 0.5[(75 - h)^2 - 75^2]$$

$$= 0.5[5625 - 150h + h^2 - 5625] = 0.5h(-150 + h)$$

$$[V(25 + h) - V(25)]/h = 0.5(-150 + h)$$

$$V'(25) = lim[h \to 0]\ 0.5(-150 + h) = -75\,m^3\,min^{-1}\,(= 1.25\,m^3\,s^{-1})$$

(c) Half-empty: V = 2500 m³:

$$5000(1 - t/100)^2 = 2500 \Rightarrow (1 - t/100)^2 = 0.5 \Rightarrow 1 - t/100 = 1/\sqrt{2}$$

$$t = 100(1 - 1/\sqrt{2}) = 100(0.2929) = 29.29\,min$$

$$V'(t) = -100(1 - t/100)\ [derived\ by\ the\ same\ limit\ method\ as\ above, general\ formula]$$

$$V'(29.29) = -100(1/\sqrt{2}) = -70.71\,m^3\,min^{-1}\,(= 1.178\,m^3\,s^{-1})$$

Worked Example 2.1.3 — Rate of Pressure Rise in a Compressed-Air Receiver

Problem: An air compressor charges a receiver. The absolute pressure (kPa) as a function of time t (s) is modelled by:

$$P(t) = 101.3 + 780(1 - e^{-0 \cdot {}^{04t}}), t \geq 0$$

(a) What is the average rate of pressure rise over the first 30 s? (b) What is the instantaneous rate at t = 0 (the initial charging rate)? (c) What physical quantity does a very large charging rate indicate in terms of compressor design?

(a) Average rate over [0, 30]:

$$P(0) = 101.3 + 780(1 - 1) = 101.3\,kPa$$

$$P(30) = 101.3 + 780(1 - e^{-1 \cdot {}^{2}}) = 101.3 + 780(1 - 0.3012) = 101.3 + 544.8 = 646.1\,kPa$$

Worked Example 2.1.3 — Rate of Pressure Rise in a Compressed-Air Receiver

$$\frac{\Delta P}{\Delta t} = (646.1 - 101.3)/30 = 544.8/30 = 18.16\ kPa\ s^{-1}$$

(b) Instantaneous rate at t = 0 (using the limit definition):

$$P(0+h) - P(0) = 780(1 - e^{-0 \cdot {}^{04h}}) - 0$$

$$[P(h) - P(0)]/h = 780(1 - e^{-0 \cdot {}^{04h}})/h$$

$$Using\ the\ standard\ limit\ lim[h \to 0](1 - e - kh)/h = k\ (proved\ by\ L'Hôpital\ or\ Taylor\ series):$$

$$P'(0) = 780 \times 0.04 = 31.2\ kPa\ s^{-1}$$

(c) A charging rate of 31.2 kPa s^{-1} = 31.2 kPa/s demands that the compressor deliver a high mass flow rate in its first stroke. This sets the **displacement volume** and **valve-plate opening area** requirements. If the rate is too low the receiver replenishment time is excessive; if too high, the mechanical loads on piston rings and crank bearings are severe.

Exercises

1. A copper wire has resistance $R(\theta) = 18.6 + 0.074\theta + 0.000043\theta^2\ (m\Omega)$, where θ is temperature (°C). (a) Compute the average rate of change of R from θ = 20 °C to θ = 80 °C. (b) Using the limit definition, find the instantaneous rate at θ = 50 °C. Give units.

2. A hydraulic actuator piston moves according to $s(t) = 3t^2 - 0.4t^3\ cm\ (0 \le t \le 5\ s)$. (a) Find the average velocity over the first 3 s. (b) Use the limit definition to find the instantaneous velocity at t = 2 s.

3. For the drainage basin in Example 2.1.2, derive a general formula for V′(t) by the limit method, and verify that V′(0) = −100 $m^3\ min^{-1}$ (the initial drain rate). Explain why the magnitude decreases with time.

4. A chemical reactor vessel has concentration $C(t) = C_0\ e - kt\ mol\ L^{-1}$. Without using differentiation rules, write out the complete limit expression for dC/dt, use the approximation $e - kh \approx 1 - kh + (kh)^2/2$ for small h, and evaluate the limit. Confirm your result gives $-kC^0e^{-kt}$.

5. State in precise physical terms why the instantaneous rate of change is more useful than the average rate for: (a) the design of a pressure-relief valve; (b) a vehicle anti-lock braking system (ABS); (c) control of a nuclear reactor's neutron flux. Write one paragraph for each.

Exercises

6. Prove algebraically that for any linear function $f(x) = mx + b$, the average rate of change over any interval $[x_1, x_2]$ equals the instantaneous rate f′(x) for every x. What does this say geometrically?

7. The fuel injection pressure (MPa) in a common-rail diesel injector rises as $P(t) = P_0(t/\tau)^2$ for $0 \le t \le \tau$. If P_0 = 180 MPa and τ = 0.8 ms, find: (a) the average pressure-rise rate over the full injection period; (b) the instantaneous rate at $t = \tau/2$; (c) the instantaneous rate at $t = \tau$.

8. A signal encoder measures angular displacement $\theta(t) = 4\,sin(10\pi t)\,rad$. Using the limit definition and the identity $sin(A + B) = sin\,A\,cos\,B + cos\,A\,sin\,B$, together with the standard $limits\ lim[h \to 0]sin(h)/h = 1\ and\ lim[h \to 0](cos(h) - 1)/h = 0$, show that $\theta'(t) = 40\pi\cos(10\pi t)$ rad s⁻¹.

2.2 Derivative from First Principles

The informal idea of the previous section is now made precise. The **derivative** of a function is the instantaneous rate of change, defined as a limit of difference quotients. This section gives the rigorous definition and then derives, without appealing to any differentiation rules, the derivatives of the six most important elementary functions.

Definition 2.2 — The Derivative

Let f be defined on an open interval containing x. The **derivative of f at x** is:

$$f'(x) = lim[h \to 0]\left[\frac{[f(x + h) - f(x)]}{h}\right]$$

provided the limit exists. When it does, f is said to be **differentiable at x**. The function that assigns f′(x) to every x at which this limit exists is called **the derivative of f** and is itself a function.

Computing a derivative directly from this definition is called differentiating **from first principles** or **ab initio**. It involves four steps: (1) substitute $x + h$ into the function; (2) form the difference $f(x + h) - f(x)$; (3) divide by h and simplify algebraically (ensuring h can be cancelled from the denominator before the limit is taken); (4) evaluate the limit as $h \to 0$. The algebraic simplification step is always the critical one, because direct substitution of h = 0 in the original quotient produces the indeterminate form 0/0.

2.2.1 Derivative of a Constant

Let $f(x) = c$. Then $f(x + h) = c$ for all h:

$\frac{[f(x+h)-f(x)]}{h} = \frac{c-c}{h} = \frac{0}{h} = 0 \; for \; all \; h \neq o$

$f'(x) = lim[h \to 0]\, 0 = 0.$

Result: $d/dx\,(c) = 0$. A constant does not change; its rate of change is zero. The graph is a horizontal line with slope zero at every point.

2.2.2 Derivative of a Power Function — The Power Rule Derived

We derive $\frac{d}{dx}(x^n)$ for any positive integer n using the **Binomial Theorem**:

$$(x + h)^n = x^n + n \cdot x^{n-1} \cdot h + ½n(n-1)x^{n-2}h^2 + \ldots + h^n$$

Forming the difference quotient:

$f(x+h) - f(x) = (x+h)^n - x^n..$

$= nx^{n-1}h + ½n(n-1)x^{n-2}h^2 + \ldots + h^n.$

$= h[nx^{n-1} + ½n(n-1)x^{n-2}h + \ldots + h^{n-1}].$

$[f(x+h) - f(x)]/h = nx^{n-1} + ½n(n-1)x^{n-2}h + \ldots + h^{n-1}.$

Every term except the first contains at least one factor of h. As h → 0, all such terms vanish:

$f'(x) = lim[h \to 0][nx^{n-1} + terms \cdot h + \ldots] = nx^{n-1}.$

Theorem 2.1 — The Power Rule

$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ for any real constant n

Although derived above for positive integers, the result holds for all real n (including fractions and negatives), as established by logarithmic differentiation in Section 3.8.

Worked derivation for n = 2: $\frac{[f(x+h)-f(x)]}{h} = \frac{[(x+h)^2-x^2]}{h} = \frac{[x^2+2xh+h^2-x^2]}{h} = \frac{2xh+h^2}{h} = 2x + h \to 2x. Hence \frac{d}{dx}(x^2) = 2x. ✓$

Worked derivation for n = 3: $\frac{[(x+h)^3-x^3]}{h} = \frac{[3x^2h+3xh^2+h^3]}{h} = 3x^2 + 3xh + h^2 \to 3x^2. Hence \frac{d}{dx}(x^3) = 3x^2. ✓$

2.2.3 Derivative of the Square Root

Let $f(x) = \sqrt{x}\,(x > 0)$. The Binomial Theorem does not apply directly. Instead we rationalise the numerator:

$$f(x+h) - f(x) = \sqrt{(x+h)} - \sqrt{x}$$

$$Multiply\ numerator\ and\ denominator\ by\ [\sqrt{(x+h)}\ +\ \sqrt{x}]:$$

$$\frac{[f(x+h)-f(x)]}{h} = \frac{\left[\sqrt{x+h}-\sqrt{x}\right]\left[\sqrt{x+h}+\sqrt{x}\right]}{\left\{h\left[\sqrt{x+h}+\sqrt{x}\right]\right\}}$$

$$= \frac{[(x+h)-\ x]}{\left\{h\left[\sqrt{x+h}+\sqrt{x}\right]\right\}} = \frac{h}{\left\{h\left[\sqrt{x+h}+\sqrt{x}\right]\right\}}$$

$$= \frac{1}{\left[\sqrt{x+h}+\ \sqrt{x}\right]}$$

$$f'(x) = \ lim[h \to 0]\left(\frac{1}{\left[\sqrt{x+h}+\sqrt{x}\right]}\right) = \frac{1}{2\sqrt{x}}$$

Result: $\boldsymbol{d/dx}\ (\sqrt{\boldsymbol{x}}) = \mathbf{1}/(\mathbf{2}\sqrt{\boldsymbol{x}}) = \left(\frac{\mathbf{1}}{\mathbf{2}}\right)\boldsymbol{x}^{-\left(\frac{1}{2}\right)}$. This matches the Power Rule with n = 1/2. ✓

2.2.4 Derivative of the Reciprocal Function

$$Let\ f(x) = \frac{1}{x}\ (x\ \neq\ 0):$$

$$f(x+h)-f(x) = \frac{1}{x+h}\ -\frac{1}{x} = \frac{[x-(x+h)]}{[x(x+h)]} = -\frac{h}{[x(x+h)]}$$

$$\frac{[f(x+h)-f(x)]}{h} = -\frac{1}{[x(x+h)]}$$

$$f'(x) = \ lim[h \to 0]\ -\frac{1}{[x(x+h)]} = -\frac{1}{x^2}$$

Result: $\frac{\boldsymbol{d}}{\boldsymbol{dx}}\ (\frac{\mathbf{1}}{\boldsymbol{x}}) = -\frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{2}}} = -\boldsymbol{x}^{-2}$. This matches the Power Rule with n = −1. ✓

2.2.5 Derivative of the Sine Function

Let $f(x) = sin\ x$. Apply the sine addition formula $sin(A+B) = sin\ A\ cos\ B\ +\ cos\ A\ sin\ B$:

$$f(x+h)\ -\ f(x) = sin(x+h)\ -\ sin\ x$$

$$= sin\ x\ cos\ h\ +\ cos\ x\ sin\ h\ -\ sin\ x$$

$$= sin\ x\ (cos\ h\ -\ 1)\ +\ cos\ x\ sin\ h$$

$$\frac{[f(x+h)-f(x)]}{h} = sin\ x \cdot \left[\frac{cos\ h-1}{h}\right] + cos\ x \cdot \left[sin\frac{h}{h}\right]$$

Two fundamental limits (proved by the squeeze theorem) are required:

$$lim[h \to 0]\left(\frac{sin\ h}{h}\right) = 1; \quad lim[h \to 0]\left(\frac{cos\ h\ -\ 1}{h}\right) = 0$$

$$f'(x) = sin\ x \cdot 0 + cos\ x \cdot 1 = cos\ x$$

Result: $\frac{d}{dx}\ (\boldsymbol{sin\ x}) = \boldsymbol{cos\ x}$**.** An entirely analogous derivation using the cosine addition formula — carried out fully in §2.2.6 below — gives $\frac{d}{dx}\ (cos\ x) = -sin\ x$.

2.2.6 Derivative of the Cosine Function

Let f(x) = cos x. Apply the cosine addition formula cos(A+B) = cos A cos B − sin A sin B:

$$f(x+h)\ -\ f(x) = cos(x+h)\ -\ cos\ x$$

$$= cos\ x\ cos\ h\ -\ sin\ x\ sin\ h\ -\ cos\ x$$

$$= cos\ x(cos\ h\ -\ 1)\ -\ sin\ x\ sin\ h$$

Dividing by h and applying the same two fundamental limits used in §2.2.5:

$$\frac{f(x+h)\ -\ f(x)}{h} = cos\ x\ \cdot \frac{cos\ h\ -\ 1}{h} - sin\ x\ \cdot \frac{sin\ h}{h}$$

$$f'(x) = cos\ x\ \cdot\ 0\ -\ sin\ x\ \cdot\ 1 = -sin\ x$$

Result: $\frac{d}{dx}\ (\boldsymbol{cos\ x}) = -\boldsymbol{sin\ x}$. Together with the result of §2.2.5, this gives the complementary pair: the derivative of each trigonometric function equals the co-function, with a sign change for the cosine. Both results are needed in oscillation analysis, AC-circuit theory, and wave propagation.

2.2.7 Derivative of the Natural Exponential Function

The number $e \approx 2.71828$ is defined uniquely by the property:

$$lim\frac{[h \to 0](e^{h}\ -\ 1)}{h} = 1$$

$$Let\ f(x) = ex:$$

$$f(x+h) - f(x) = e^{x+h}\ -\ e^{x} = e^{x}(e^{h} - 1)$$

$$\frac{[f(x+h) - f(x)]}{h} = e^{x} \cdot \frac{e^{h} - 1}{h}$$

$$f'(x) = lim[h \to 0]\ e^{x} \cdot \frac{e^{h} - 1}{h} = e^{x} \cdot 1 = e^{x}$$

Result: $\frac{d}{dx}\,(e^x) = e^x$. The natural exponential is the unique non-trivial function equal to its own derivative. This characterises exponential growth and decay throughout all of engineering dynamics.

2.2.8 Derivative of the Natural Logarithm

$$Let\ f(x) = ln\,x\ (x > 0):$$

$$f(x+h) - f(x) = ln(x+h) - ln\,x = ln[\frac{x+h}{x}] = ln(1 + \frac{h}{x})$$

$$\frac{[f(x+h) - f(x)]}{h} = (\frac{1}{h}\ ln(1 + \frac{h}{x})$$

$$Let\ u = \frac{h}{x}\ so\ h = ux, h \to 0 \Leftrightarrow u \to 0:$$

$$= \left(\frac{1}{ux}\right) \cdot ln(1+u) = (\frac{1}{x}) \cdot (\frac{1}{u}) \cdot ln(1+u) = (\frac{1}{x}) \cdot ln(1+u)^{\mathrm{n)u}}$$

Using the standard definition $lim[u \to 0]\ \left(\frac{(1+u)1}{u}\right) = e, so\ ln(1+u)\left(\frac{1}{u}\right) \to ln\,e = 1$:

$$f'(x) = \left(\frac{1}{x}\right) \cdot 1 = \frac{1}{x}$$

Result: $\frac{d}{dx}\,(\boldsymbol{ln\,x}) = \frac{1}{x}$. This result underlies logarithmic differentiation, entropy calculations, decibel scales, and any engineering context involving multiplicative or ratio-based processes.

2.2.9 Higher-Order Derivatives

Since $f'(x)$ is itself a function, it may be differentiated to yield the **second derivative**

$$f''(x) = \frac{d}{dx}[f'(x)], and\ so\ on:$$

$$f'(x) — first\ derivative\ (velocity, temperature\ gradient, \ldots)$$

$$f''(x) — second\ derivative\ (acceleration, curvature, \ldots)$$

$$f'''(x) — third\ derivative\ (jerk, rate\ of\ bending, \ldots)$$

$$f^{\mathrm{n}}(x) — n - th\ derivative$$

For $f(x) = x4: f' = 4x^3, f'' = 12x^2, f''' = 24x, f(4) = 24, f(5) = 0$. All derivatives beyond the 4th are zero — a consequence of the Power Rule applied repeatedly.

Worked Example 2.2.1 — First Principles: General Cubic Polynomial

Problem: Differentiate $f(x) = 3x^3 - 5x^2 + 8x - 4$ from first principles.

$$f(x+h) = 3(x+h)^3 - 5(x+h)^2 + 8(x+h) - 4$$

$$= 3(x^3 + 3x^2h + 3xh^2 + h^3) - 5(x^2 + 2xh + h^2) + 8x + 8h - 4$$

$$= 3x^3 + 9x^2h + 9xh^2 + 3h^3 - 5x^2 - 10xh - 5h^2 + 8x + 8h - 4$$

$$f(x+h) - f(x) = 9x^2h + 9xh^2 + 3h^3 - 10xh - 5h^2 + 8h$$

$$= h(9x^2 + 9xh + 3h^2 - 10x - 5h + 8)$$

$$\frac{[f(x+h) - f(x)]}{h} = 9x^2 + 9xh + 3h^2 - 10x - 5h + 8$$

$$f'(x) = lim[h \to 0](9x^2 + 9xh + 3h^2 - 10x - 5h + 8) = 9x^2 - 10x + 8$$

Answer: $\boldsymbol{f'(x) = 9x^2 - 10x + 8}$. Verification by Power Rule: $\frac{d}{dx}[3x^3] = 9x^2$, $\frac{d}{dx}[-5x^2] = -10x$, $\frac{d}{dx}[8x] = 8$, $\frac{d}{dx}[-4] = 0$. ✓

Worked Example 2.2.2 — First Principles: Rational Function $\frac{1}{2x+1}$

Problem: Differentiate $f(x) = \frac{1}{2x+1}$ from first principles.

$$f(x+h) = \frac{1}{2(x+h)+1} = \frac{1}{2x+2h+1}$$

$$f(x+h) - f(x) = \frac{1}{2x+2h+1} - \frac{1}{2x+1}$$

$$= \frac{[(2x+1) - (2x+2h+1)]}{[(2x+2h+1)(2x+1)]}$$

$$= -\frac{2h}{[(2x+2h+1)(2x+1)]}$$

$$\frac{[f(x+h) - f(x)]}{h} = -\frac{2}{[(2x+2h+1)(2x+1)]}$$

$$f'(x) = lim[h \to 0] -\frac{2}{[(2x+2h+1)(2x+1)]} = -\frac{2}{(2x+1)^2}$$

Answer: $\boldsymbol{f'(x) = -2/(2x+1)^2}$. Verification (Chain Rule, Section 3): $\frac{d}{dx}[(2x+1)-1] = -1 \cdot (2x+1) - 2 \cdot 2 = -\frac{2}{(2x+1)^2}$. ✓

Worked Example 2.2.3 — First Principles: cos x

Problem: $Prove \frac{d}{dx}\ (cos\ x)\ =\ -sin\ x$ from first principles.

$$f(x+h) - f(x)\ =\ cos(x+h)\ -\ cos\ x$$

$$=\ cos\ x\ cos\ h\ -\ sin\ x\ sin\ h\ -\ cos\ x\ [cosine\ addition\ formula]$$

$$=\ cos\ x\ (cos\ h - 1)\ -\ sin\ x\ sin\ h$$

$$\frac{[f(x+h) - f(x)]}{h}\ =\ cos\ x \cdot \left[\frac{cos\ h - 1}{h}\right]\ -\ sin\ x \cdot \left[sin\frac{h}{h}\right]$$

$$f'(x)\ =\ cos\ x \cdot 0\ -\ sin\ x \cdot 1\ =\ -sin\ x\ \checkmark$$

Worked Example 2.2.4 — Second Derivative and Elastic Bar

Problem: An elastic bar of length L is fixed at one end (x = 0) and loaded by a uniformly distributed axial load. Its axial displacement is $u(x)\ =\ (\frac{w}{2EA})(Lx\ -\frac{x^2}{2})$ (where w = load intensity, E = Young's modulus, A = cross-section area). Find u′(x) (the axial strain) and $u''(x), and\ interpret\ both.$

$$u'(x)\ =\ (\frac{w}{2EA})(L\ -\ x)\ [axial\ strain;\ dimensionless\ if\ consistent\ units]$$

$$u''(x)\ =\ (\frac{w}{2EA})(-1)\ =\ -\frac{w}{2EA}\ =\ constant$$

The strain $u'(x)$ decreases linearly from a maximum of $wL/(2EA)$ at the fixed end ($x = 0$) to zero at the free end ($x = L$). The constant second derivative confirms a **uniform rate of strain decay** along the bar. The axial stress at any section is $\sigma(x)\ =\ E \cdot u'(x)\ =\ w(L - x)/(2A)$, confirming that stress is maximum at the fixed support — consistent with the reaction force carrying the full distributed load.

Exercises

1. Differentiate $f(x)\ =\ 4x^2\ -\ 7x\ +\ 3$ from first principles and verify with the Power Rule.

2. Using the limit definition, find $d/dx(\sqrt{(5x)})$ and state its domain.

3. Differentiate $f(x)\ =\ 1/(x^2)$ from first principles. [Hint: form $f(x+h) - f(x)$, find a common denominator, factor h, then cancel.]

Exercises

4. Prove from first principles that $d/dx(e^{2x}) = 2e^{2x}$. [Hint: write $e^{2(x+h)} = e^{2x} \cdot e^{2h}$, let $u = 2h$, and use $\lim[u \to 0](e^u - 1)/u = 1$.]

5. Find d^2y/dx^2 for $y = 5x^3 - 12x^2 + 7x - 2$ by applying the first-principles method twice. Verify your answer using the Power Rule.

6. Differentiate $f(x) = x^2 \sin x$ from first principles. [Hint: $f(x+h) = (x+h)^2 \sin(x+h)$; expand and group terms carefully.]

7. The deflection of a cantilever beam tip is $\delta(P) = PL^3/(3EI)$ (metres), where P is the tip load (N). Using the limit definition, find $d\delta/dP$ and interpret it as a flexibility coefficient with correct units.

8. Find the third derivative of $f(x) = \ln x$ by differentiating successively from the result $d/dx(\ln x) = 1/x$. Give a general formula for $f^{(n)}(x)$ for any positive integer n.

9. Verify numerically that the derivative of $\sin x$ at $x = \pi/6$ predicted by the formula $d/dx(\sin x) = \cos x$ is approximately equal to $[\sin(\pi/6 + 0.001) - \sin(\pi/6)]/0.001$. How accurate is the approximation?

10. A thermoelectric generator has output voltage $V(T) = \alpha(T - T_0) + \beta(T - T_0)^2\ (mV)$, where $T_0 = 300\ K, \alpha = 0.050\ mV\ K - 1, \beta = 0.0002\ mV\ K^{-2}$. Find $V'(T)$ from first principles and evaluate at T = 350 K. What does $V'(T)$ represent physically?

2.3 Differentiability and Continuity

Two fundamental properties of functions — continuity and differentiability — are often confused. This section establishes their precise relationship: differentiability is the **stronger** condition. Every differentiable function is continuous, but a continuous function need not be differentiable.

2.3.1 The Theorem: Differentiability Implies Continuity

Theorem 2.2 — Differentiability Implies Continuity

If f is differentiable at x = a, then f is continuous at x = a.

Proof:

$$We\ must\ show\ lim[x \to a]\ f(x) = f(a), i.e., lim[h \to 0][f(a+h) - f(a)] = 0. Write:$$

$$f(a+h) - f(a) = \left\{\frac{[f(a+h) - f(a)]}{h}\right\} \cdot h$$

$$Taking\ limits\ and\ using\ the\ product\ rule\ for\ limits:$$

$$lim[h \to 0][f(a+h) - f(a)] \;=\; lim[h \to 0]\{\frac{[f(a+h) - f(a)]}{h}\} \cdot lim[h \to 0]\, h$$

$$= f'(a) \cdot 0 \;=\; 0$$

$Therefore\ lim[h \to 0]\, f(a+h) \;=\; f(a), which\ is\ precisely\ continuity\ at\ a.$ ■

Contrapositive: if f is **not** continuous at a, then it is automatically not differentiable there. All jump discontinuities, removable holes, and infinite asymptotes preclude differentiability immediately.

2.3.2 The Converse Fails: Continuous but Not Differentiable

The function $f(x) = |x|$ is the canonical counterexample. It is continuous everywhere, including at x = 0. But the limit defining the derivative at x = 0 fails because the left-hand and right-hand limits are different:

Example 2.3.1 — The Absolute Value: Continuous but Not Differentiable at $x = 0$

Left-hand derivative (h < 0):

$$lim\frac{[h \to 0^-][|0+h| - |0|]}{h} \;=\; lim\frac{[h \to 0^-]|h|}{h}$$

$$Since\ h \;<\; 0: |h| = \;-h, so\frac{|h|}{h} \;=\; -\frac{h}{h} \;=\; -1$$

$$Left - hand\ derivative \;=\; -1$$

Right-hand derivative (h > 0):

$$lim[h \to 0^+]\left(\frac{|h|}{h}\right) = \frac{lim[h \to 0^+]h}{h} \;=\; 1$$

$$Right - hand\ derivative \;=\; +1$$

$-1 \neq +1$, so the two-sided limit does not exist. Therefore $f(x) = |x|$ is **not differentiable at x = 0**. The graph has a sharp **corner** at the origin. Geometrically there is no unique tangent line at that point: approaching from the left the tangent has slope −1, from the right slope +1.

2.3.3 Classification of Non-Differentiable Points

Points where a continuous function fails to be differentiable are classified into three types:

(a) Corner Points (Kinks)

Both one-sided derivatives exist but are unequal. Example: $f(x) = |x|\ at\ x = 0$ (slopes −1 and +1). In engineering, corners arise in piecewise-linear stress-strain models, kinked structural members, and load-distribution diagrams.

(b) Cusp Points

Both one-sided derivatives are infinite with opposite signs. Example: $f(x) = x^{\left(\frac{2}{3}\right)}\ at\ x =$ 0 where $\frac{d}{dx}\left(x^{\left(\frac{2}{3}\right)}\right) = \left(\frac{2}{3}\right)x^{-\frac{1}{3}} \rightarrow \pm\infty$. The graph comes to a sharp pointed tip. Cusps appear in crack-tip stress-field models in fracture mechanics.

(c) Vertical Tangent Points

The derivative grows without bound but with the same sign from both sides. Example: $f(x) = x^{\left(\frac{1}{3}\right)}\ at\ x = 0$ where $d/dx(x^{1/3}) = (1/3)x^{-2/3} \rightarrow +\infty$. The curve is still continuous but momentarily vertical. Vertical tangents occur in inversion mechanisms and in free-surface flow profiles near critical depth.

Worked Example 2.3.2 — piecewise Function: Checking Differentiability

Problem: The current delivered by a switched-mode power supply is modelled by:

$$i(t) = \{\, 6t^2, 0 \leq t < 1.5\ s$$

$$\{\, 27 - 12t, 1.5 \leq t \leq 2.25\ s$$

Is i(t) continuous and differentiable at t = 1.5 s? Interpret any failure of differentiability in circuit terms.

Continuity:

$$lim[t \rightarrow 1.5^{-}]\ 6t^2 = 6(1.5)^2 = 13.5\ A$$

$$lim[t \rightarrow 1.5^{+}]\ (27 - 12t) = 27 - 18 = 9\ A$$

Left limit (13.5) ≠ Right limit (9) ⇒ i(t) is NOT continuous at t = 1.5 s

Since the function is not even continuous at t = 1.5 s, it is certainly not differentiable there. Physically, the current has a **jump discontinuity** at t = 1.5 s. In the physical circuit this would require an infinite voltage across any inductance L in the line ($V = L\frac{di}{dt} \rightarrow \infty$), which is physically impossible and indicates that the piecewise model must be refined to include a short transition region.

Worked Example 2.3.3 — Finding Constants for Differentiability

Problem: The piecewise displacement function of a cam follower is:

Worked Example 2.3.2 — piecewise Function: Checking Differentiability

$$s(t) = \{ at^2 + b, t < 2\, s$$

$$\{ 12t - 8, t \geq 2\, s$$

Find constants a and b such that s(t) is differentiable at t = 2 s.

Continuity at t = 2 (necessary condition):

$$a(2)^2 + b = 12(2) - 8 \Rightarrow 4a + b = 16 \ldots (i)$$

$$\boldsymbol{Equal\ derivatives\ at\ t = 2}:$$

$$s'(t)\ on\ left = 2at;\ at\ t = 2: 2a(2) = 4a$$

$$s'(t)\ on\ right = 12;\ at\ t = 2: 12$$

$$For\ differentiability: 4a = 12 \Rightarrow a = 3$$

$$Substituting\ a = 3\ into\ (i): 12 + b = 16 \Rightarrow b = 4.$$

Answer: a = 3, b = 4. The follower acceleration on the left segment is 2a = 6 m s^{-2}, matching the acceleration implied by the right-hand linear segment (which has zero acceleration since it is linear). Wait — a linear function has zero second derivative, while the left branch at t = 2 gives s″ = 2a = 6. These are unequal, meaning s″ is **discontinuous** at t = 2 (a kink in the velocity). The displacement and velocity are both continuous and equal, making s differentiable, but the acceleration has a jump. In cam design this jump is called a **kinematic shock** and generates impulsive forces.

Exercises

1. Show that $f(x) = |3x - 6|$ is not differentiable at $x = 2$. Find the left and right-hand derivatives there and draw a sketch showing the corner.

2. Prove rigorously that f(x) = $x^{1/3}$ is continuous at x = 0 but has a vertical tangent there (and hence is not differentiable in the ordinary sense). Find the one-sided limits of [f(h)–f(0)]/h as h→0^+ and h→0^-.

3. The piecewise stress-strain relationship for a bilinear material is $\sigma(\varepsilon) = \{E\varepsilon, \varepsilon < \varepsilon_y\}$ and $\{\sigma_y + E_t(\varepsilon - \varepsilon_y), \varepsilon \geq \varepsilon_y\}$. Show that σ is continuous but not differentiable at $\varepsilon = \varepsilon_y\ if\ E_t \neq E$. What does the failure of differentiability represent physically?

4. Find constants p and q such that f(x) = {px^3 + q, x ≤ 1} and {$4x^2$ − 1, x > 1} is differentiable everywhere.

Exercises

5. The beam deflection at a section with an internal hinge is $\delta(x) = \{Ax(L-x),\ 0 \le x \le L/2\}$ and $\{B(L-x)^2,\ L/2 < x \le L\}$. For what values of A and B is δ continuous at $x = L/2$? Is it possible to choose A and B so that δ is also differentiable there?

6. Prove that if f and g are both differentiable at x = a, then so is f + g, and $(f+g)'(a) = f'(a) + g'(a)$. [Use the limit definition and properties of limits.]

2.4 Geometric Interpretation: Tangents and Normal

The derivative $f'(a)$ is the slope of the **tangent line** to the curve y = f(x) at the point P = (a, f(a)). This geometric interpretation is not merely a pretty picture: it is the foundation of linear approximation, Newton's root-finding method, the analysis of cam mechanisms, and the formulation of boundary conditions in contact problems.

2.4.1 The Tangent Line

Equation of the Tangent Line at x = a

$$y - f(a) = f'(a) \cdot (x - a)$$

This is the unique straight line through (a, f(a)) with slope $f'(a)$. For x near a it provides a **local linear approximation** to f:

$$f(x) \approx f(a) + f'(a)(x - a)\ for\ x \approx a$$

The tangent line is the limiting position of secant lines through (a, f(a)) as the second point approaches (a, f(a)) along the curve. If $f'(a) > 0$ the tangent rises to the right (f is locally increasing); if $f'(a) < 0$ it falls (f is locally decreasing); if $f'(a) = 0$ the tangent is horizontal (a candidate turning point or stationary point).

2.4.2 The Normal Line

Equation of the Normal Line at x = a

The **normal** to $y = f(x)\ at\ (a, f(a))$ is the line perpendicular to the tangent. Since perpendicular lines have slopes whose product is −1:

$$y - f(a) = \left[-\frac{1}{f'(a)}\right] \cdot (x - a)\ (when\ f'(a) \neq 0)$$

Special cases: if $f'(a) = 0$ (horizontal tangent) the normal is vertical: x = a. If $f'(a)$ is undefined (vertical tangent) the normal is horizontal: y = f(a).

Normal lines are critical in mechanical design: the **pressure angle** in cam-follower design is the angle between the contact force (always along the normal) and the follower motion direction. ISO standards specify that the pressure angle must not exceed 30° for

ordinary cams to prevent follower jamming. Normal lines also define the direction of heat flux across surfaces (Fourier's Law), the direction of stress resultants on inclined planes, and the angle of incidence in optical and antenna systems.

2.4.3 Angle of Inclination

The **angle of inclination** α of the tangent to y = f(x) at x = a is the angle measured anti-clockwise from the positive x-axis:

$\alpha = \arctan(f'(a)),\ \alpha \in [0°, 180°)$

In beam theory the **slope angle** θ(x) = arctan(δ′(x)) of the elastic curve is the angle through which each cross-section rotates. For small deflections tanθ ≈ θ, so θ(x) ≈ δ′(x) (in radians) — the slope of the deflection curve equals the rotation of the section.

2.4.4 Angle of Intersection of Two Curves

If two curves intersect at a point and their tangent slopes there are m_1 and m_2, the acute angle θ between them satisfies:

$$\tan\theta = |m_1 - m_2|/|1 + m^1 m^2|$$

If $m_1 m_2 = -1$ the curves are **orthogonal** (θ = 90°). Orthogonal families are pervasive in physics: streamlines and equipotentials in potential flow are mutually orthogonal; isotherms and heat-flux lines are orthogonal; electric field lines and equipotential surfaces are orthogonal. This geometric fact is a consequence of the gradient being perpendicular to level curves (Chapter 6).

Worked Example 2.4.1 — Tangent and Normal to a Turbine Nozzle Profile

Problem: A turbine nozzle vane profile is described by $y = 5 + 2x - x^2$. Find the tangent and normal at x = 1 and determine the pressure angle (angle between normal and vertical).

$$y(1) = 5 + 2(1) - (1)^2 = 6. Point\ P = (1, 6).$$

$$\frac{dy}{dx} = 2 - 2x. At\ x = 1: f'(1) = 2 - 2 = 0$$

The tangent slope is zero ⇒ **horizontal tangent** at (1, 6): equation y = 6.

The normal slope is undefined (vertical): equation x = 1.

Pressure angle: the normal is vertical (90° from horizontal). The angle from vertical is 0°. The cam follower rises vertically through the apex of this nozzle profile — zero pressure angle, ideal mechanical advantage. This is why nozzle profiles are shaped to present a flat vertex to the flow.

Worked Example 2.4.2 — Tangent, Normal, and Inclination: pipe Bend Profile

Problem: A pipe-bend transition piece is designed with centreline profile $y = x^3/3 - x$. At the point where $x = 2\,m$: (a) find the tangent and normal equations; (b) find the angle of inclination of the tangent; (c) find where the normal crosses the x-axis (relevant for locating the centre of curvature).

$$y(2) = 8/3 - 2 = 8/3 - 6/3 = 2/3. Point\ P = (2, 2/3).$$

$y' = x^2 - 1. At\ x = 2: slope\ m_t = 4 - 1 = 3.$

(a) Tangent:

$$y - \frac{2}{3} = 3(x-2) \Rightarrow y = 3x - 6 + \frac{2}{3} \Rightarrow y = 3x - \frac{16}{3}$$

Normal (slope = −1/3):

$$y - \frac{2}{3} = -\frac{1}{3(x-2)} \Rightarrow y = -\frac{x}{3} + \frac{2}{3} + \frac{2}{3} \Rightarrow y = -\frac{x}{3} + \frac{4}{3}$$

(b) Angle of inclination:

$$\alpha = arctan(3) \approx 71.6°$$

(c) x-intercept of normal (y = 0):

$$0 = -\frac{x}{3} + \frac{4}{3} \Rightarrow x = 4\,m$$

The normal to the pipe centreline at x = 2 m passes through (4, 0). The radius of curvature at this point is the distance from

$$P = (2, \frac{2}{3})\ to\ (4,0): r = \sqrt{[(4-2)^2 + (0 - \frac{2}{3})^2]} = \sqrt{[4 + \frac{4}{9}]} = \sqrt{(\frac{40}{9})} \approx 2.11\,m.$$

Worked Example 2.4.3 — Orthogonal Curves: Thermal and Flow Fields

Problem: Two families of curves are defined by: C_1: $x^2 + 2y^2 = k$ (ellipses, representing isotherms in an anisotropic medium) and C_2: $y = mx^2$ (parabolas, candidate heat-flux paths). Show that these two families are orthogonal at every intersection.

Slope of C_1 by implicit differentiation:

$$2x + 4y \cdot \left(\frac{dy}{dx}\right) = 0 \Rightarrow m^1 = \frac{dy}{dx} = -\frac{x}{2y}$$

Slope of C_2:

Worked Example 2.4.3 — Orthogonal Curves: Thermal and Flow Fields

$$m^2 = \frac{dy}{dx} = 2mx$$

Product of slopes:

$$m_1 \cdot m_2 = [-\frac{x}{2y}] \cdot [2mx] = -\frac{mx^2}{y}$$

At an intersection of C_1 and C_2, $y = mx^2$, so $mx^2/y = mx^2/(mx^2) = 1$. Therefore $m_1m_2 = -1$. **The curves are orthogonal at every intersection.** ✓ The parabolas $y = mx^2$ are indeed the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 = k$. In the thermal context, heat flows along these parabolas from hot ellipse to cold ellipse.

Worked Example 2.4.4 — Parabolic Antenna: Tangent and Focal Property

Problem: A parabolic dish antenna has cross-sectional profile $y = x^2/(4f)$ *where* $f = 0.8\ m$ is the focal length. At the rim point $x = 1.2\ m$: (a) find the tangent slope; (b) show the normal passes through the focus (0, f).

$$y(1.2) = \frac{(1.2)^2}{4 \times 0.8} = \frac{1.44}{3.2} = 0.45\ m. Point\ P = (1.2, 0.45).$$

$$y' = \frac{2x}{4f} = \frac{x}{2f}. At\ x = 1.2: m_t = \frac{1.2}{2 \times 0.8} = 0.75.$$

(a) $Tangent\ slope = 0.75$. Tangent: $y - 0.45 = 0.75(x - 1.2)$.

(b) $Normal\ slope = -1/0.75 = -4/3$. Normal: $y - 0.45 = -4/3(x - 1.2)$.

Test whether (0, f) = (0, 0.8) lies on the normal:

$$RHS: -\frac{4}{3}(0 - 1.2) + 0.45 = -\frac{4}{3}(-1.2) + 0.45 = 4 \times \frac{1.2}{3} + 0.45 = 1.6 + 0.45 = \ldots$$

$$Wait — recalculate: -\frac{4}{3}(0 - 1.2) = -\frac{4}{3} \times (-1.2) = +1.6;\ 1.6 + 0.45 = 2.05 \neq 0.8$$

Let us verify via the general formula. For parabola $y = x^2/(4f)$, the normal at $(x_0, {x_0}^2/(4f))$ has equation:

$$y - {x_0}^2/(4f) = -(2f/x_0)(x - x_0)$$

$$At\ x = 0: y = {x_0}^2/(4f) + -(2f/x_0)(0 - x_0) = {x_0}^2/(4f) + 2f$$

For the normal to pass through $(0, f)$: ${x_0}^2/(4f) + 2f = f \Rightarrow {x_0}^2/(4f) = -f$. This has no real solution, meaning **not every normal passes through the focus**. Only the axial ray ($x_0 = 0$) does. The parabolic reflector works because incoming parallel rays

Worked Example 2.4.4 — Parabolic Antenna: Tangent and Focal Property

reflect off each point and **all reflected rays** pass through the focus — this is the **optical property of the parabola**, proved via angle of incidence = angle of reflection using the tangent slope, not the normal intercept.

Exercises

1. Find the tangent and normal to $y = x^3 - 6x + 2$ *at the point* $(2, -2)$.

2. A heat-sink fin has profile $y = 8 - 3x^2$ (mm). Find: (a) the tangent at x = 1 mm; (b) the normal at x = 1 mm; (c) the angle of inclination of the tangent; (d) where the normal crosses the x-axis.

3. Find all points on the curve $y = x^3 - 3x^2 + 2$ where the tangent line is: (a) horizontal; (b) parallel to the line $y = 6x - 1$.

4. Prove that the curves $y = e^x$ and $y = e^{-x}$ are orthogonal at their point of intersection (0, 1) by showing the product of their slopes there is −1.

5. The cam profile $y = \sqrt{(r^2 - x^2)}$ is a semicircle of radius r. Show that the normal to this curve at any point (x_0, y_0) on the semicircle passes through the origin. What is the engineering significance of this result for cam-follower design?

6. Two stress trajectories in a plane-stress field are described by $y = Ax^2$ (maximum principal stress lines) and $y^2 = Bx$ (minimum principal stress lines, parabolas). Show they are orthogonal at every intersection by finding the product of their slopes at a general intersection point.

7. Find constants A and B such that the parabola $y = Ax^2 + B$ is tangent to the line $y = 3x - 2$ *at the point* $(2, 4)$. Verify your answer.

8. A Bezier path for a robot end-effector is $y = x(4 - x)/4$ (m). At x = 1 m, find the unit tangent vector and unit normal vector. The feed force is applied along the tangent direction — what angle does it make with the horizontal?

2.5 Physical Interpretation

The derivative was invented to solve physical problems. This section develops the two most important classes of physical interpretation: the kinematics of particles and mechanisms (velocity, acceleration, and jerk) and the rates that govern thermodynamic processes (Fourier's Law, Newton's Law of Cooling, and gas-process rates).

2.5.1 Velocity, Acceleration, and Jerk

Kinematic Derivatives

For a particle with position s(t) along a line:

$$\boldsymbol{v(t)} \;=\; ds/dt \;=\; s'(t)\;[velocity, m\,s^{-1}]$$

$$\boldsymbol{a(t)} \;=\; dv/dt \;=\; d^2s/dt^2 \;=\; s''(t)\;[acceleration, m\,s^{-2}]$$

$$\boldsymbol{j(t)} \;=\; da/dt \;=\; d^3s/dt^3 \;=\; s'''(t)\;[jerk, m\,s^{-3}]$$

Sign convention: positive s is a chosen reference direction. A positive v means motion in that direction; negative v means the opposite. $Speed = |v|$. Positive a means v is increasing; negative a means v is decreasing (decelerating if also v > 0). The particle is at rest when v = 0; it reverses direction if v changes sign there.

The third kinematic derivative, ***jerk*** $j(t) = s'''(t)$, measures how rapidly acceleration changes. It is directly perceptible to human occupants of vehicles — the sensation of being "pulled back into your seat" on a fast train involves a large positive jerk. Railway and elevator standards typically limit jerk to 2 m s^{-3} for passenger comfort. In machine-tool path planning, high jerk excites mechanical resonances and causes dimensional errors; CNC controllers use jerk-limited trajectory profiles called **S-curves** to smooth the motion.

2.5.2 Locating Extremes of Velocity

Velocity is maximised or minimised when its rate of change is zero, i.e., when a(t) = v′(t) = 0. At such an instant the acceleration changes sign (or equals zero throughout, implying constant velocity). In mechanism design, the maximum piston velocity in a slider-crank mechanism occurs at roughly 90° crank angle, as confirmed by differentiating the velocity and setting it to zero.

Worked Example 2.5.1 — Full Kinematic Analysis: Slider-Crank Piston

Problem: The displacement (m) of a piston in a single-cylinder engine from the top dead centre (TDC) position is modelled by:

$$s(t) \;=\; r[1 \;-\; cos(\omega t)] \;+\; (\frac{r^2}{4L})[1 \;-\; cos(2\omega t)]$$

$where\; r \;=\; 0.048\,m\;(crank\;radius), L \;=\; 0.192\,m\;(connecting-rod\;length, giving\; L/r \;=\; 4), and\; \omega \;=\; 220\pi\;rad\;s-1\;(engine\;speed \;=\; 6600\,rpm)$. Derive v(t) and a(t) by differentiating. Evaluate all three at $t \;=\; 0\;(TDC)\; and\; at\; \omega t \;=\; \pi/2\;(crank\; at\; 90°)$.

Velocity:

Worked Example 2.5.1 — Full Kinematic Analysis: Slider-Crank Piston

$$v(t) = \frac{ds}{dt} = r\omega\, sin(\omega t) + (\frac{r^2}{4L}) \cdot 2\omega\, sin(2\omega t)$$

$$= r\omega\, sin(\omega t) + (\frac{r^2\omega}{2L})\, sin(2\omega t)$$

$$With\, r = 0.048, \omega = 220\pi, L = 0.192, \frac{r}{L} = 0.25:$$

$$v(t) = 0.048 \times 220\pi \cdot sin(\omega t) + 0.048^2 \times \frac{220\pi}{2 \times 0.192} \cdot sin(2\omega t)$$

$$= 10.56\pi \cdot sin(\omega t) + 0.264\pi \cdot sin(2\omega t)\, m\, s^{-1}$$

Acceleration:

$$a(t) = \frac{dv}{dt} = r\omega^2\, cos(\omega t) + (\frac{r^2\omega^2}{L})\, cos(2\omega t)$$

$$= 0.048 \times (220\pi)^2\, cos(\omega t) + 0.048^2 \times \frac{(220\pi)^2}{0.192} \cdot cos(2\omega t)$$

$$\approx 22\,747\, cos(\omega t) + 5\,687\, cos(2\omega t)\, m\, s^{-2}$$

At TDC ($\omega t = 0$):

$$s(0) = 0\, m\, (reference), v(0) = 0\, m\, s^{-1}\, (piston\, momentarily\, at\, rest\, \checkmark)$$

$$a(0) = 22\,747 + 5\,687 = 28\,434\, m\, s^{-2} \approx 2899\, g$$

The piston acceleration at TDC is nearly 2900 times gravitational acceleration — a reminder of why high-speed engine pistons must be ultralight aluminium or titanium alloys.

At 90° ($\omega t = \pi/2$):

$$v = 10.56\pi \cdot sin(\frac{\pi}{2}) + 0.264\pi \cdot sin(\pi) = 10.56\pi + 0 = 33.18\, m\, s^{-1}$$

$$a = 22\,747\, cos(\frac{\pi}{2}) + 5\,687\, cos(\pi) = 0 - 5\,687 = -5\,687\, m\, s^{-2}$$

At 90° the piston moves at its near-maximum velocity of 33.18 m s⁻¹ and is already decelerating at 5687 m s⁻²

Worked Example 2.5.2 — Jerk in a Servo-Actuator Motion Profile

Problem: An electric linear servo moves a pick-and-place arm according to:

Worked Example 2.5.2 — Jerk in a Servo-Actuator Motion Profile

$$s(t) = A\,t^3 - B\,t^2\,(0 \le t \le T)$$

where A = 4 mm s⁻³ and B = 18 mm s⁻² and T = 3 s. (a) Find v(t), a(t), and j(t). (b) At what time does the arm reach maximum velocity? (c) Evaluate the jerk at t = 0 and t = T. Comment on the suitability of this profile for precision work.

(a) Derivatives:

$$v(t) = 3At^2 - 2Bt = 12t^2 - 36t\ mm\ s^{-1}$$

$$a(t) = 6At - 2B = 24t - 36\ mm\ s^{-2}$$

$$j(t) = 6A = 24\ mm\ s^{-3}\ (constant!)$$

(b) Maximum velocity when a(t) = 0:

$$24t - 36 = 0 \Rightarrow t = 1.5\ s = \frac{T}{2}$$

$$v(1.5) = 12(1.5)^2 - 36(1.5) = 27 - 54 = -27\ mm\ s^{-1}\ (negative \rightarrow maximum\ speed\ in\ backward\ direction)$$

(c) Jerk:

$j(0) = j(T) = 24\ mm\ s^{-3}$ *at all times (constant jerk profile)*

A constant (non-zero) jerk means the acceleration changes linearly with time — this is the **linear ramp** profile. It is smoother than a trapezoidal velocity profile (which has instantaneous jerk spikes) but less smooth than an S-curve (which has bounded jerk and zero jerk at start/end). The constant jerk of 24 mm s⁻³ is well within the 2000 mm s⁻³ guideline for precision servo systems. However, since jerk is non-zero at t = 0 and t = T, the profile causes an impulsive load change at the endpoints unless blending curves are added.

2.5.3 Rates in Thermodynamics

Fourier's Law of Heat Conduction

When a temperature distribution T(x) exists along a one-dimensional conductor of thermal conductivity k (W m⁻¹ K⁻¹), the heat flux (W m⁻²) is:

$$q = -k\frac{dT}{dx}$$

The negative sign expresses the Second Law: heat flows from hot to cold, i.e., in the direction of **decreasing** temperature (opposite to the positive dT/dx direction). In a composite wall the temperature gradient is different in each layer, inversely proportional

to its conductivity, so highly conductive metals have nearly uniform temperatures while insulation layers carry steep gradients.

Newton's Law of Cooling

When a body at temperature T is exposed to a fluid at constant ambient temperature T_∞, the rate of heat loss from the body surface is proportional to the temperature excess:

$$\frac{dT}{dt} = -k(T - T\infty), k > 0$$

This is a **first-order linear ODE** whose solution is $T(t) = T_\infty + (T_0 - T_\infty)e^{-kt}$, where $T_0 = T(0)$. The derivative provides the instantaneous cooling rate — essential for quench-bath design in heat treatment, cold-chain logistics, and electronics thermal management.

Ideal Gas Rate Equations

Differentiating the ideal gas law PV = nRT with respect to time gives the general rate equation:

$$V\frac{dP}{dt} + P\frac{dV}{dt} = nR\frac{dT}{dt}$$

Setting individual terms to zero yields the special process rates: isothermal $\left(T = const, \frac{dT}{dt} = 0\right): \frac{dP}{dt} = -(\frac{P}{V})\left(\frac{dV}{dt}\right)$; isobaric (P = const): dV/dt = (nR/P)dT/dt; isochoric (V = const): dP/dt = (nR/V)dT/dt.

Adiabatic Process Rate

For an adiabatic process $PV\gamma = C$, differentiating with respect to t:

$$\left(\frac{dP}{dt}\right) \cdot V\gamma + P\gamma V\gamma^{-1}\left(\frac{dV}{dt}\right) = 0 \Rightarrow \frac{dP}{dt} = -\gamma(\frac{P}{V})\left(\frac{dV}{dt}\right)$$

Worked Example 2.5.3 — Fourier's Law: Composite Furnace Wall

Problem: A furnace wall consists of two layers: firebrick (k_1 = 1.1 W m^{-1} K^{-1}, thickness L_1 = 0.25 m) and mineral wool insulation (k_2 = 0.038 W m^{-1}, thickness L_2 = 0.10 m). Inner surface at T_1 = 1100 °C, outer surface at T_3 = 30 °C. At steady state the heat flux q is the same through both layers. Find: (a) the interface temperature T_2; (b) dT/dx in each layer; (c) q.

Thermal resistances:

$$R_1 = L_1/k_1 = 0.25/1.1 = 0.2273\ m^2\ K\ W^{-1}$$

$$R_2 = L_2/k_2 = 0.10/0.038 = 2.632\ m^2\ K\ W^{-1}$$

$$R_{total} = 0.2273 + 2.632 = 2.859\ m^2\ K\ W^{-1}$$

Worked Example 2.5.3 — Fourier's Law: Composite Furnace Wall

(c) Heat flux:

$$q = \frac{T^1 - T^3}{R_{total}} = \frac{1100 - 30}{2.859} = \frac{1070}{2.859} = 374.3\ W\ m^{-2}$$

(a) Interface temperature:

$$q = (T_1 - T_2)/R_1 \Rightarrow T_2 = T_1 - q \cdot R_1 = 1100 - 374.3 \times 0.2273 = 1100 - 85.1 = 1014.9\ °C$$

(b) Temperature gradients:

$$Layer\ 1: \frac{dT}{dx} = -q/k_1 = -374.3/1.1 = -340.3\ °C\ m^{-1}$$

$$Layer\ 2: \frac{dT}{dx} = -q/k_2 = -374.3/0.038 = -9850\ °C\ m^{-1}$$

The temperature gradient in the insulation layer is **nearly 29 times larger** than in the firebrick, despite the same heat flux. This shows graphically why insulation with low k is effective: it concentrates the temperature drop where it does not damage the structure. The total temperature drop across the insulation is 9850 × 0.10 = 985 °C (from 1015 °C to 30 °C), confirming that almost all the resistance is in the mineral wool layer.

Worked Example 2.5.4 — Newton's Law of Cooling: Steel Quenching

Problem: A steel billet leaves a furnace at 860 °C and is quenched in an agitated oil bath maintained at 45 °C. The Newton cooling constant is k = 0.022 s^{-1}. (a) Write T(t). (b) Find dT/dt at t = 0 and t = 45 s. (c) Find the time to reach the martensite-start temperature M_s = 320 °C and the cooling rate at that instant.

$$T(t) = 45 + (860 - 45)e^{-0.022t} = 45 + 815e^{-0.022t}\ °C$$

(b) Cooling rates:

$$\frac{dT}{dt} = -0.022 \times 815 \cdot e^{-0.022t} = -17.93\ e^{-0.022t}\ °C\ s^{-1}$$

$$t = 0: \frac{dT}{dt} = -17.93\ °C\ s^{-1}\ (17.93\ °C\ per\ second)$$

$$t = 45: \frac{dT}{dt} = -17.93\ e^{-0.99} = -17.93 \times 0.3716 = -6.66\ °C\ s^{-1}$$

(c) Time to M_s = 320 °C:

Worked Example 2.5.4 — Newton's Law of Cooling: Steel Quenching

$$45 + 815e^{-0.022\,t} = 320 \Rightarrow e^{-0.022\,t} = \frac{275}{815} = 0.33742$$

$$-0.022t = ln(0.33742) = -1.0866 \Rightarrow t = 49.4\,s$$

$$\frac{dT}{dt}|_{\{t=49.4\}} = -17.93 \times 0.33742 = -6.05\,°C\,s^{-1}$$

The billet must cool through the martensite-start temperature (320 °C) at 6.05 °C s^{-1} at t = 49.4 s. If the cooling rate at M_s is too slow (< about 1.5 °C s^{-1} for some steels) diffusive transformations form pearlite instead of martensite and the required hardness will not be achieved. The present rate of 6.05 °C s^{-1} is well above this critical cooling rate, confirming the oil quench is adequate for through-hardening this billet.

Worked Example 2.5.5 — Adiabatic Compression Rate in a Diesel Cylinder

Problem: During the compression stroke of a diesel engine the gas behaves adiabatically ($PV^{1.4} = C$). At the instant when $V = 6.0 \times 10^{-4}$ m^3 and the piston is moving upward at $\frac{dV}{dt} = -0.18\,m^3s-1, the\ pressure\ is\ P = 280\,kPa. Find\,\frac{dP}{dt}$.

$$\frac{dP}{dt} = -\gamma(\frac{P}{V})dV/dt = -1.4 \times (280\,000 / 6.0 \times 10^{-4}) \times (-0.18)$$

$$= -1.4 \times 4.667 \times 10^{8} \times (-0.18)$$

$$= +1.173 \times 10^{8}\,Pa\,s^{-1} = 117.3\,MPa\,s^{-1}$$

The cylinder pressure is rising at **117 MPa per second**. This extreme rate of pressure increase is why diesel engine components must be manufactured to tight tolerances with high fatigue resistance. It also explains the characteristic diesel knock — the rapid pressure rise produces a sharp impulsive force on the piston crown, connecting rod, and crankshaft.

Exercises

1. A spacecraft has vertical position $s(t) = 15t^2 - 0.9t^3\,km$ (t in minutes, 0 ≤ t ≤ 11 min). Find: (a) v(t) and a(t); (b) the time of maximum velocity; (c) the maximum height reached; (d) the jerk j(t).

2. A rotating shaft has angular displacement $\theta(t) = 6t^3 - t^4\,rad$ $(0 \le t \le 3\,s)$. Find the angular velocity ω(t), angular acceleration α(t), and jerk. At what time(s) is the shaft momentarily at angular rest?

Exercises

3. An electromagnetic actuator displaces a valve plate by $x(t) = X_0[1 - cos(\pi t/T)]\ for\ 0 \le t \le T$. Show that the velocity is zero at $t = 0\ and\ t = T$ (soft stop), that the maximum velocity is $\pi X_0/T$, and that the jerk at t = 0 is zero. Why is zero jerk at start and end important for valve design?

4. A steel rod of diameter 20 mm and length 400 mm is heated from 20 °C to 420 °C. The temperature along its length (m) at a particular instant is $T(x) = 20 + 1000x - 1250x^2$. If $k = 50\ W\ m^{-1}\ K^{-1}$, find the heat flux at x = 0, x = 0.2 m, and x = 0.4 m. At which location is the heat flux zero and what does this mean?

5. An aluminium casting ($T_\infty = 25$ °C, $k = 0.019\ s^{-1}$) cools from 620 °C. Find: (a) T(t); (b) the cooling rate at t = 0; (c) the time when T = 100 °C; (d) the cooling rate at that time; (e) the time for the cooling rate to drop below 1 °C s^{-1}.

6. Air at 300 K expands isothermally (T = const) in a piston-cylinder. Initially $V_0 = 5 \times 10^{-4}\ m^3$ and $P_0 = 200$ kPa. If the piston moves outward so that V increases at $3 \times 10^{-5}\ m^3\ s^{-1}$, find dP/dt at the initial instant using the ideal gas law result.

7. The temperature at the surface of a re-entry vehicle varies as $T(t) = T_{max} \sin(\pi t/(2t_m))$, $0 \le t \le t_m$. Find the maximum rate of temperature rise and the time at which it occurs. If $T_{max} = 1800$ °C and $t_m = 120$ s, evaluate the maximum rate in °C s^{-1}.

8. Prove that Newton's Law of Cooling $T' = -k(T - T_\infty)$ is satisfied by $T(t) = T_\infty + (T_0 - T_\infty)e^{-kt}$ by direct substitution and differentiation. Confirm every step with correct units.

2.5.4 Rates in Electrical Engineering

Electrical quantities change continuously with time, and their rates of change govern the behaviour of every reactive circuit element. The derivative is the natural language of transient circuit analysis.

Capacitor current—voltage relationship. A capacitor of capacitance C (farads) stores charge Q = CV. The current flowing into the capacitor equals the rate of change of charge:

$$i(t) = \frac{dQ}{dt} = C\,\frac{dV}{dt} \quad [A]$$

A capacitor passes current only when its voltage is changing. If V is constant (DC steady state), dV/dt = 0 and i = 0 — the capacitor is an open circuit to DC. Conversely, an instantaneous voltage jump would require infinite current, which is physically impossible: capacitor voltages cannot change instantaneously. This derivative constraint is fundamental to filter design, energy storage, and power-electronics switching analysis.

Inductor voltage—current relationship. An inductor of inductance L (henrys) stores energy in its magnetic field. The voltage across it is proportional to the rate of change of current:

$$V(t) = L\,\frac{di}{dt} \quad [V]$$

A large di/dt — produced by suddenly opening a switch in an inductive circuit — generates a large voltage spike (V = L di/dt → ∞ if di/dt → ∞). This is the physical basis of the inductive flyback spike that destroys transistors and relay contacts if not suppressed by a freewheeling diode. Inductor currents also cannot change instantaneously: an instantaneous current step would require infinite voltage, again physically impossible.

Power rate and energy. The instantaneous power delivered to any circuit element is p(t) = V(t) i(t) (watts). The energy stored or dissipated from time t_1 to t_2 is the integral of power, but the instantaneous rate at which energy changes is the power itself:

$$\frac{dW}{dt} = p(t) = V(t)\,i(t) \quad [W = J\,s^{-1}]$$

Dimensional summary. The derivative rule [dy/dx] = [y]/[x] immediately yields:

$$[dV/dt] = V\,s^{-1}, \quad [di/dt] = A\,s^{-1}, \quad [d^2V/dt^2] = V\,s^{-2}$$

Worked Example 2.5.6 — RC Circuit Charging: Voltage and Current Rates

Problem: A series RC circuit (R = 10 kΩ, C = 47 μF) is connected to a 12 V DC supply at t = 0. The capacitor voltage is V(t) = 12(1 − e^{−t/τ}) V, where τ = RC = 0.47 s. Find: (a) the charging current i(t); (b) the initial charging rate dV/dt at t = 0; (c) the time constant τ in terms of the derivative; (d) the power dissipated in R at t = 0.

(a) Charging current:

$$i(t) = C\,\frac{dV}{dt} = 47\times10^{-6}\times\frac{12}{\tau}\,e^{-t}/\tau = 1.2\times10^{-3}\,e^{-t}/\tau\,A = 1.2\,mA\times e^{-t}/\tau$$

(b) Initial charging rate:

$$\frac{dV}{dt}\Big|_0 = \frac{12}{0.47} = 25.5\,V\,s^{-1}$$

The capacitor charges at 25.5 V per second initially, decaying exponentially as the voltage approaches the supply.

(c) Time constant interpretation: τ = V_{teah} / (dV/dt)|0 = 12 / 25.5 = 0.47 s ✓ The time constant equals the reciprocal of the initial normalised charging rate.

(d) Initial power in R:

$$p(0) = i(0)^2 R = (1.2 \times 10^{-3})^2 \times 10^4 = 14.4\,mW$$

All supply energy at t = 0 is dissipated in R; as V→12 V, i→0 and power dissipation falls to zero — correctly modelling the approach to DC steady state.

2.6 Notation and Units in Engineering Contexts

Calculus is written in several equivalent but syntactically distinct notations, each with its own strengths and natural domain. An engineer reading papers from different disciplines, or working with international standards, must recognise and translate between all of them instantly. A notation error — confusing a total derivative with a partial derivative, or misreading the order of a higher-order derivative — can lead to a fundamentally incorrect equation and a dangerous design.

2.6.1 Leibniz Notation

Introduced by Gottfried Wilhelm Leibniz in the 1670s, this notation writes the derivative of y with respect to x as:

$$\left[\frac{dy}{dx}\right] \ (read: "dee\ y\ by\ dee\ x"\ or\ "the\ derivative\ of\ y\ with\ respect\ to\ x")$$

Higher derivatives:

$$\left[\frac{d^2y}{dx^2}\right] (second\ derivative) \left[\frac{d^3y}{dx^3}\right] \ (third) \left[\frac{d^{\mathrm{n}}y}{dx^{\mathrm{n}}}\right] (n-th)$$

The evaluation at a specific point x = a is written:

$$\frac{dy}{dx}\Big|_{\{x=a\}} or \left[\frac{dy}{dx}\right]_{\{x=a\}}$$

Leibniz notation dominates in thermodynamics ($dU/dT, dP/dV, dS/dT$), fluid mechanics (du/dx, dp/dt), electrical engineering ($dV/dt, di/dt$), and chemical engineering (dC/dt, dn/dt). Its great advantage is that the variable of differentiation is always explicit, and the chain rule looks like ordinary fraction cancellation:

$$\frac{dy}{dt} = (\frac{dy}{du})(\frac{du}{dt})$$

This “fraction-like” property is not accidental; it follows from the limit definition and is made rigorous by the theory of differentials.

2.6.2 Newton's Dot Notation

Isaac Newton used superscript dots for time derivatives exclusively:

$$\dot{x} = \frac{dx}{dt}\ (velocity), \qquad \ddot{x} = \frac{d^2x}{dt^2}(acceleration), \qquad \dddot{x} = \frac{d^3x}{dt^3}\ \ (jerk)$$

Newton's notation is the standard in classical mechanics, structural dynamics, and control engineering:

- Newton's Second Law: $F = m\ddot{x} = m \cdot d^2x/dt^2$
- Spring-mass-damper: $m\ddot{x} + c\dot{x} + kx = F(t)$
- Euler-Lagrange equation: $\frac{d}{dt}(\partial L/\partial \dot{x}) - \partial L/\partial x = 0$

The notation is concise for time derivatives but is limited to that case and becomes awkward for derivatives of order four or higher (the fourth time-derivative, jounce, would be $\ddot{\dot{x}}$ and is rarely written in dot notation beyond three). For such cases the parenthetical superscript notation $x^{(4)}$ is preferred.

2.6.3 Lagrange's Prime Notation

Joseph-Louis Lagrange introduced the prime notation, now universal in pure and applied mathematics:

$$f'(x)\ (first), f''(x)\ (second), f'''(x)\ (third), f^{n}(x)\ (n-th)$$

Prime notation is compact and the most common in textbook derivations, differential equations, and elasticity:

- Beam deflection: $EI\delta''''(x) = q(x)$
- ODE: $y'' + 3y' + 2y = 0$
- Slope of elastic curve: $\theta(x) = \delta'(x)$

Its weakness is that the variable of differentiation must be inferred from context. When a function of multiple variables is written as f(x) this is not a problem, but in problems involving both time and position variables the notation becomes potentially ambiguous unless carefully defined at first use.

2.6.4 Euler's Operator Notation

Leonhard Euler introduced the linear operator D:

$$Df = \frac{df}{dx}, \qquad D^2f = \frac{d^2f}{dx^2}, \qquad D^{n}f = \frac{d^{n}f}{dx^{n}}$$

Operator notation is the most powerful for solving linear ODEs with constant coefficients, because D obeys polynomial algebra. The equation $(D^2 + 5D + 6)y = e^{2x}$ can be factored as $(D+2)(D+3)y = e^{2x}$ and solved by the operator-inverse method (the D-operator or operational calculus). This technique is standard in electrical circuit analysis and vibration theory.

2.6.5 Partial Derivative Notation

When a function depends on two or more variables, the partial derivative with respect to one variable (all others held constant) is denoted with the curly ∂ symbol:

∂f/∂x (partial of f with respect to x, all other variables fixed)

This notation is **essential** and must never be confused with d/dx, which implies differentiation with total dependence on x (including any implicit dependence of other variables on x). For a function T(x, y, z, t) of position and time:

- $\partial T/\partial t$ = rate of change of temperature at a fixed spatial point (Eulerian rate)
- $\partial T/\partial x$ = temperature gradient in the x-direction (at fixed y, z, t)

In thermodynamics, the partial derivative of internal energy U(T, V) with respect to temperature at constant volume is written $(\partial U/\partial T)_V$. The subscript indicates what is held constant — a convention that prevents ambiguity when the same function can be differentiated holding different variables fixed.

2.6.6 Summary Table of Notations

Notation	1st Deriv.	2nd Deriv.	n-th Deriv.	Primary Engineering Use
Leibniz	dy/dx	d^2y/dx^2	d^ny/dx^n	Thermo, fluid mech., electrical eng.
Newton	$\dot{x}$	$\ddot{x}$	x^n	Dynamics, control systems, Lagrangian mech.
Lagrange	$f'(x)$	$f''(x)$	$f^n(x)$	Pure math, ODEs, elasticity, beam theory
Euler	Df	D^2f	D^nf	Operator methods, linear ODE systems
Partial	$\partial f/\partial x$	$\partial^2 f/\partial x^2$	$\partial^n f/\partial x^n$	Heat equation, fluid flow, EM fields, stress analysis

2.6.7 Dimensional Analysis of Derivatives

The dimensional rule is universal:

$$[d^n y/dx^n] = [y] / [x]^n$$

Applications across engineering domains:

• $[d^2\delta/dx^2]$ = m/m² = m⁻¹ (beam curvature, related to bending moment $M = EI\kappa, \kappa = d^2\delta/dx^2$)

• $[d^3\delta/dx^3]$ = m/m³ = m⁻² (related to shear force $Q = EI \cdot d^3\delta/dx^3$)

• $[d^4\delta/dx^4]$ = m⁻³ (related to distributed load intensity $w = EI \cdot d^4\delta/dx^4$ in N m⁻¹)

• $[dV/dt]$ = V s⁻¹ (rate of voltage change across capacitor: $i = C \cdot dV/dt$ in amperes)

• $[d\rho/dt]$ = kg m⁻³ s⁻¹ (mass accumulation rate in the continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$)

Dimensional consistency is the engineer's most reliable check. If a differentiation step produces a result with wrong units, there is certainly an error in the algebra.

2.6.8 ISO, IEEE, and ASME Notation Conventions

Three major standards bodies specify preferred notations:

• **ISO 80000 (Quantities and Units):** recommends Leibniz notation for derivatives; specifies that physical quantity symbols be set in *italic* type (e.g., m for mass, T for temperature) while units are set in roman upright type (e.g., m for metres, K for kelvin).

• **IEEE Standards (Electrical & Electronic):** uses dot notation for time derivatives of circuit variables ($\dot{\varphi}$ for flux linkage rate, $\dot{m}$ for charge rate = current); uses j (not i) for imaginary unit to avoid confusion with current i.

• **ASME Standards (Mechanical):** employs Leibniz notation for stress analysis and heat transfer; uses prime notation for beam deflection problems; uses subscripts to distinguish partial from total differentiation (e.g., u_x for ∂u/∂x in solid mechanics).

Best practices for engineering documents: (1) define all notation symbols at first use; (2) maintain consistency throughout the document; (3) use subscripts in thermodynamics to show what is held constant; (4) always carry units through differentiation steps when numerical values are involved.

Worked Example 2.6.1 — Converting Between All Four Notations

Problem: The governing equation for torsional vibration of a drill string is written in Newton notation as: $I\theta + c\dot{u} + k\theta = M_0 \cos(\Omega t)$

Here θ(t) is the twist angle, I = polar mass moment of inertia, c = damping coefficient, k = torsional stiffness, M₀ = excitation torque amplitude. Rewrite this in (a) Leibniz notation; (b) Lagrange notation; (c) Euler operator notation. State the order and classify the equation.

Worked Example 2.6.1 — Converting Between All Four Notations

(a) Leibniz:

$$I \cdot d^2\theta/dt^2 \; + \; c \cdot d\theta/dt \; + \; k\theta \; = \; M_0 \, cos(\Omega t)$$

(b) Lagrange:

$$I\theta''(t) \; + \; c\theta'(t) \; + \; k\theta(t) \; = \; M_0 \, cos(\Omega t)$$

(c) Euler operator:

$$(ID^2 \; + \; cD \; + \; k)\theta \; = \; M_0 \, cos(\Omega t)$$

This is a **second-order, linear, ordinary differential equation with constant coefficients** and a sinusoidal forcing function (non-homogeneous). The order is 2 (highest derivative is $d^2\theta/dt^2$). Dimensional check: [Iu̧] = kg m² · rad s⁻² = N m (torque); similarly [cu̧] = N m s · s⁻¹ = N m; [kθ] = N m rad⁻¹ · rad = N m. All terms have units of N m, confirming consistency.

Worked Example 2.6.2 — Dimensional Analysis of the Beam Equation

Problem: The Euler-Bernoulli beam equation is $EI \, d^4\delta/dx^4 = q(x)$, where δ is deflection (m), x is position (m), E is Young's modulus (Pa = N m⁻²), I is second moment of area (m⁴), and q(x) is distributed load intensity (N m⁻¹). Verify dimensional consistency of the equation.

$$[EI \, d^4\delta/dx^4] \; = \; [E] \cdot [I] \cdot [\delta]/[x]^4$$

$$= \; (N \, m^{-2}) \cdot (m^4) \cdot (m)/(m^4)$$

$$= \; N \, m^{-2} \cdot m^4 \cdot m \cdot m^{-4}$$

$$= \; N \, m^{-2} \cdot m \; = \; N \, m^{-1}$$

$$[q(x)] \; = \; N \, m^{-1} \; \checkmark$$

The dimensional analysis confirms that the beam equation $EI \, d^4\delta/dx^4 \; = \; q$ has units N m⁻¹ on both sides. The fourth-order derivative reduces the units of the deflection (m) by a factor m⁻⁴, and the EI prefactor restores the balance. This is why engineers working with beam equations must use consistent units throughout — mixing metres and millimetres in the same equation will give answers wrong by factors of 10^9 to 10^{12}.

Worked Example 2.6.3 — Partial vs. Total Derivative in Gas Dynamics

Problem: The pressure in a converging nozzle is a function of both position x (m) and time t (s): P(x, t) = P₀ e^{−αx} sin(ωt). (a) Find the partial derivative ∂P/∂t (rate of pressure

Worked Example 2.6.3 — Partial vs. Total Derivative in Gas Dynamics

change at a fixed location). (b) If a sensor moves with the flow at speed u (m s^{-1}), so that x = ut, find the total rate dP/dt experienced by the moving sensor, using the chain rule.

(a) Partial derivative (fixed x):

$$\partial P/\partial t = P_o\, e^{-\alpha x} \cdot \omega \cos(\omega t)$$

(b) Total derivative (sensor moving at x = ut):

$$dP/dt = \partial P/\partial t + (\partial P/\partial x)\cdot(dx/dt)$$

$$\partial P/\partial x = P_o(-\alpha)e^{-\alpha x} \sin(\omega t)$$

$$dx/dt = u \text{ (sensor speed)}$$

$$dP/dt = P_o\, e^{-\alpha x}[\omega \cos(\omega t) - \alpha u \sin(\omega t)]$$

The sensor experiences an additional pressure rate $-\alpha u\, P_0\, e^{-\alpha x}\sin(\omega t)$ due to its movement through the spatially varying pressure field. This **convective** contribution would be invisible to a stationary transducer. In gas dynamics this is the distinction between the Eulerian rate (∂P/∂t, fixed point) and the Lagrangian rate (dP/dt, following a fluid parcel) — the foundation of the material derivative concept (Chapter 6).

Exercises

1. Rewrite the beam deflection equation $EI\, d^4\delta/dx^4 = q(x)$ in: (a) Lagrange prime notation with independent variable x; (b) Euler operator notation. What is the order of this equation?

2. The damped wave equation is $\partial^2 u/\partial t^2 + 2\beta\partial u/\partial t = c^2\, \partial^2 u/\partial x^2$. Identify whether each term uses a total or partial derivative. If u represents acoustic pressure (Pa), t is time (s), x is distance (m), and c is wave speed (m s^{-1}), verify that every term has the same units.

3. A capacitor voltage satisfies $V(t) = V_0(1 - e^{-t/RC})$. Write dV/dt and d^2V/dt^2 in all four notations (Leibniz, Newton, Lagrange, Euler). Identify the units of each derivative.

4. The heat flux through a wall with position-dependent conductivity is $q = -k(x)\cdot dT/dx$. If $k(x) = k_0(1 + \beta x)$ and $T(x) = T_1 - (T_1 - T_2)\cdot x/L$ (linear profile), find q(x) and verify units.

5. Show that for z = f(x, y) where y is also a function of x, the total derivative $dz/dx \neq \partial z/\partial x$ in general. Give a specific example with f(x,y) = xy and $y = x^2$ to illustrate the difference.

6. A thermocouple measures gas temperature in a turbine. The gas temperature field is $T(r, t) = T_0 + \Delta T\, e^{-r/R}\cos(\omega t)$ (K), where r is radius from axis (m). Write ∂T/∂r (radial

Exercises

gradient at fixed time) and $\partial T/\partial t$ (temporal rate at fixed position) in Leibniz notation and give their units.

Chapter 2 Summary

This chapter has built the concept of differentiation from the ground up, progressing from physical motivation through rigorous definition to geometric, physical, and notational facets.

Section 2.1 established that the average rate of change of f over an interval is the slope of the secant line: $\Delta y/\Delta x = [f(x+\Delta x)-f(x)]/\Delta x$. The instantaneous rate of change at a point is the limit of this ratio as $\Delta x \to 0$, and it equals the slope of the tangent line at that point. Units follow the rule $[\Delta y/\Delta x] = [y]/[x]$.

Section 2.2 gave the formal definition of the derivative and derived, from first principles, all six fundamental results:

$$d/dx(c) = 0; \quad d/dx(x^n) = nx^{n-1}; \quad d/dx(\sqrt{x}) = 1/(2\sqrt{x})$$

$$d/dx(1/x) = -1/x^2; \quad d/dx(\sin x) = \cos x; \quad d/dx(\cos x) = -\sin x$$

$$d/dx(e^x) = e^x; \quad d/dx(\ln x) = 1/x$$

The Power Rule $d/dx(x^n) = nx^{n-1}$ was derived via the Binomial Theorem for all positive integers and extended to all real n. Higher-order derivatives are obtained by successive differentiation.

Section 2.3 proved that differentiability implies continuity (Theorem 2.2) but that the converse fails. Non-differentiable points of continuous functions are classified as corners, cusps, or vertical tangents. Engineering consequences include impulsive forces from kinematic shocks at corners in cam profiles and infinite voltages implied by jump discontinuities in current.

Section 2.4 interpreted $f'(a)$ as the slope of the tangent $y - f(a) = f'(a)(x-a)$. The normal has slope $-1/f'(a)$. The angle of inclination is $\alpha = \arctan(f'(a))$. Two curves are orthogonal at an intersection if $m_1 m_2 = -1$. Applications include cam pressure-angle analysis, parabolic reflector optics, and orthogonal trajectory families in potential-flow and heat-transfer analysis.

Section 2.5 applied derivatives to kinematics ($v = s'$, $a = s''$, $j = s'''$), thermodynamics, and electrical engineering. Fourier's Law $q = -k\,dT/dx$, Newton's Law of Cooling $dT/dt = -k(T-T_\infty)$, and adiabatic process rate $dP/dt = -\gamma(P/V)dV/dt$ were all derived or confirmed. § 2.5.4 extended this to electrical engineering: the capacitor relation $i = C\,dV/dt$ and the inductor relation $V = L\,di/dt$ both derive from the same derivative

concept and carry the same dimensional logic using the derivative concept. Each carries precise units that provide a constant check on correctness.

Section 2.6 surveyed four notational systems — Leibniz (dy/dx), Newton ($\dot{x}$), Lagrange (f'), Euler (D) — plus partial-derivative notation ($\partial/\partial x$). The dimensional rule $[d^n y/dx^n] = [y]/[x]^n$ was stated and illustrated across beam theory, kinematics, and electrics. ISO, IEEE, and ASME conventions were summarised.

Together these ideas form the conceptual foundation for all of Chapter 3 (differentiation rules), Chapter 4 (advanced differentiation techniques), and Chapter 5 (higher-order derivatives). Every technique introduced later in the book is a direct extension of the limit-based definition developed here.

CHAPTER 3

Fundamental Rules of Differentiation

Introduction: The Indispensable Language of Change

The **derivative** of a function at a point is the instantaneous rate at which the function's output is changing with respect to its input at that instant. Conceptually it is the slope of the tangent line to the graph at that point. Computationally it is defined as the limit of the difference quotient:

The Limit (First-Principles) Definition

$$f'(x) = \lim_{h \to 0} \frac{[f(x + h) - f(x)]}{h}$$

This definition is the rigorous foundation of everything that follows. However, applying it directly to every function we encounter would be slow and error-prone. The central contribution of this chapter is a compact set of **differentiation rules** — each derived once and for all from the limit definition — which allow us to differentiate virtually any elementary function with mechanical efficiency.

The power of these rules cannot be overstated. With fewer than ten rules, all of which fit on a single page, a trained engineer can differentiate functions that would take pages of direct limit calculations. The rules of differentiation are to calculus what the multiplication tables are to arithmetic: a compact encoding of structure that turns hard computations into routine ones.

This chapter is structured so that:

1. **Every rule is derived, not just stated.** Derivations reveal the assumptions underlying each rule and guard against misapplication.
2. **Every rule is illustrated with multiple worked examples** of increasing complexity, beginning with simple cases and progressing to mixed combinations and engineering contexts.
3. **Every major result is cross-verified.** Numerical central-difference checks confirm analytical derivatives, providing a built-in error-detection mechanism.

4. **Engineering applications are woven throughout,** demonstrating that these are not abstract mathematical exercises but practical tools used daily in electrical, mechanical, civil, and chemical engineering.

Historical Note: Newton, Leibniz, and the Calculus

Differential calculus was developed independently in the late 17th century by Isaac Newton (1642–1726) and Gottfried Wilhelm Leibniz (1646–1716). Newton called his approach 'the method of fluxions', conceiving the derivative as a 'fluxion' — the rate of flow of a quantity he called a 'fluent'. Leibniz developed the now-standard notation dy/dx and d/dx, which makes the chain rule and substitution arguments almost self-evident. The Product Rule, Quotient Rule, and Chain Rule were all known to both inventors, though their formulation and notation differed. The notations and rules we use today are essentially those of Leibniz, refined and made rigorous by Cauchy, Weierstrass, and others in the 19th century through the ε-δ formulation of limits.

3.1 Basic Rules of Differentiation

We begin with four rules that together allow differentiation of any polynomial or combination of power-law terms. They are derived directly from the limit definition, and each is completely rigorous.

3.1.1 The Constant Rule

A constant function $f(x) = c$ for all x has a perfectly flat graph — its slope is zero at every point. No matter how x changes, the output stays fixed, so the rate of change is zero. Let us verify this formally.

Derivation from First Principles

Proof: Constant Rule

$$f'(x) = lim_{\{h\to 0\}} \frac{[f(x+h) - f(x)]}{h} = lim_{\{h\to 0\}} \frac{[c - c]}{h} = lim_{\{h\to 0\}} 0 = 0$$

Constant Rule: $\frac{d}{dx}[c] = 0$ (c = any real constant)

Remark: The graphical meaning is transparent: the graph of y = c is a horizontal line, and horizontal lines have slope zero. No matter what value of x you choose, the tangent to y = c is the line itself — with slope zero.

3.1.2 The Power Rule

The Power Rule is the backbone of polynomial and algebraic differentiation. It states that to differentiate a power of x, bring the exponent down as a multiplier and reduce the exponent by one.

Derivation for n ∈ ℕ (via Binomial Theorem)

Let $f(x) = xn$, n a positive integer. Expanding $(x+h)^n$ by the Binomial Theorem:

Proof: Power Rule (n ∈ ℕ)

$$(x+h)^{\mathrm{n}} = x^{\mathrm{n}} + C(n,1)\cdot x^{\mathrm{n}-1}\cdot h + C(n,2)\cdot x^{\mathrm{n}-2}\cdot h^2 + \cdots + h^{\mathrm{n}}$$

$$f'(x) = lim_{\{h\to 0\}} \frac{[(x^{\mathrm{n}} + n\cdot x^{\mathrm{n}-1}h + C(n,2)x^{\mathrm{n}-2}h^2 + \cdots + h^{\mathrm{n}}) - x^{\mathrm{n}}]}{h}$$

$$= lim_{\{h\to 0\}} [n\cdot x^{\mathrm{n}-1} + C(n,2)\cdot x^{\mathrm{n}-2}\cdot h + \cdots + h^{\mathrm{n}-1}]$$

$$= n\cdot x^{\mathrm{n}-1}\ (all\ remaining\ terms\ contain\ h\ and\ vanish)$$

Power Rule: $\frac{d}{dx}[x^{\mathrm{n}}] = n\cdot x^{\mathrm{n}-1}\ (n\in\mathbb{R}, x\neq 0\ if\ n<0)$

Extension to rational and real n: For n = p/q (rational), implicit differentiation of $y^q = x^p$ gives $qy^q - 1\cdot y' = px^p - 1, so\ y' = \left(\frac{p}{q}\right)x^{\frac{p}{q}} - 1$ — the same formula. For irrational n, logarithmic differentiation (Section 3.8) confirms the rule. In all cases, the formula $\frac{d}{dx}[x^n] = nx^{n-1}$ holds for all real n ≠ 0.

Power Rule — Quick Reference Table

Function	***Derivative***	***Simplified***
$\boldsymbol{c}$ (***constant***)	0	0
$\boldsymbol{x}$	1	1
$\boldsymbol{x^2}$	$2x$	2
$\boldsymbol{x^3}$	$3x^2$	$3x^2$
$\boldsymbol{x^{\mathrm{n}}}$	$n\cdot x^{\mathrm{n}-1}$	$n\cdot x^{\mathrm{n}-1}$
$\boldsymbol{\sqrt{x} = x^{\frac{1}{2}}}$	$\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$	$1/(2\sqrt{x})$
$\boldsymbol{\sqrt[3]{x} = x^{\frac{1}{3}}}$	$\left(\frac{1}{3}\right)x^{-\frac{2}{3}}$	$1/(3\sqrt[3]{x^2})$
$\boldsymbol{1/x = x^{-1}}$	$-x^{-2}$	$-1/x^2$
$\boldsymbol{1/x^2 = x\text{^}(-2)}$	$-2x^{-3}$	$-2/x^3$

$x^{\frac{3}{2}}$	$\left(\frac{3}{2}\right)x^{\frac{1}{2}}$	$(3/2)\sqrt{x}$
$x^{-\frac{3}{4}}$	$\left(-\frac{3}{4}\right)x^{-\frac{7}{4}}$	$-\frac{3}{4x^{\frac{7}{4}}}$

Example 1: Systematic Power Rule Application

Differentiate each expression:

(a) $f(x) = x^{12} \rightarrow n = 12 \rightarrow f'(x) = 12x^{11}$

(b) $g(x) = x^{\frac{5}{4}} \rightarrow n = \frac{5}{4} \rightarrow g'(x) = \left(\frac{5}{4}\right)x^{\frac{1}{4}} = \left(\frac{5}{4}\right) \cdot \sqrt[4]{x}$

(c) $h(x) = x^{-7} \rightarrow n = -7 \rightarrow h'(x) = -7x^{-8} = -7/x^8$

(d) $k(x) = \sqrt[5]{(x^3)} = x^{\frac{3}{5}} \rightarrow n = \frac{3}{5} \rightarrow k'(x) = \left(\frac{3}{5}\right)x^{-\frac{2}{5}} = 3/(5x^{\frac{2}{5}})$

Example 2: Rewrite Before Differentiating

Differentiate f(x) = $(3x^4 - 5x^2 + 2) / x^3$.

Step 1 — Divide: $f(x) = 3x - 5x^{-1} + 2x^{-3}$

Step 2 — Power Rule: $f'(x) = 3 + 5x^{-2} - 6x^{-4} = 3 + 5/x^2 - 6/x^4$

Answer: $f'(x) = 3 + 5/x^2 - 6/x^4$

3.1.3 The Sum and Difference Rule

Differentiation distributes over addition and subtraction. This makes it a *linear operator* — one of its most important structural properties, which it shares with integration, Fourier transforms, and Laplace transforms in engineering analysis.

Derivation

Proof: Sum and Difference Rules

$$Let\ S(x) = f(x) + g(x). Then:$$

$$S'(x) = lim_{\{h \to 0\}} \frac{\left[\left(f(x+h) + g(x+h)\right) - \left(f(x) + g(x)\right)\right]}{h}$$

$$= lim_{\{h \to 0\}} [f(x+h) - f(x)]/h + lim_{\{h \to 0\}} [g(x+h) - g(x)]/h$$

$$= f'(x) + g'(x)$$

(The same argument gives $[f - g]' = f' - g'$)

Sum Rule: $\frac{d}{dx}\,[f(x)\,+\,g(x)]\,=\,f'(x)\,+\,g'(x)$

Difference Rule: $\frac{d}{dx}\,[f(x)\,-\,g(x)]\,=\,f'(x)\,-\,g'(x)$

3.1.4 The Scalar Multiple Rule

Proof: Scalar Multiple Rule

Let $M(x)\,=\,k\cdot f(x)$, k constant. Then:

$$M'(x)=\,lim_{\{h\to 0\}}\,k\cdot\frac{[f(x+h)-f(x)]}{h}\,=\,k\,\cdot\,lim_{\{h\to 0\}}\frac{[f(x+h)-f(x)]}{h}\,=\,k\cdot f'(x)$$

$$Scalar\ Multiple\ Rule\text{:}\frac{d}{dx}\,[k\cdot f(x)]\,=\,k\,\cdot\,f'(x)\,(k\,=\,constant)$$

Warning: k must be a **constant**. If the multiplier depends on x, the **Product Rule** (Section 3.2) must be used instead. A common error is applying the Scalar Multiple Rule when the 'constant' is actually a function of x.

3.1.5 Combining the Basic Rules — Differentiating Polynomials and Algebraic Expressions

The four basic rules work together seamlessly. For any polynomial or linear combination of power-law terms, we differentiate term by term, bringing constants through as scalar multiples and applying the power rule to each term. The sum/difference rule then combines the results.

Example 3: Complete Polynomial Differentiation

Differentiate $p(x)\,=\,9x^8\,-\,4x^6\,+\,7x^4\,-\,3x^3\,+\,x^2\,-\,8x\,+\,5.$

Term by term: $p'(x)\,=\,72x^7\,-\,24x^5\,+\,28x^3\,-\,9x^2\,+\,2x\,-\,8$

Answer: $p'(x)\,=\,72x^7\,-\,24x^5\,+\,28x^3\,-\,9x^2\,+\,2x\,-\,8$

Example 4: Mixed Power Expression

Differentiate $f(x)\,=\,5x^{\frac{3}{2}}\,-\,8x^{-\frac{1}{2}}\,+\,4x^{-3}\,-\,2\sqrt{x}\,+\,6.$

Rewrite: $f(x)\,=\,5x^{\frac{3}{2}}\,-\,8x^{-\frac{1}{2}}\,+\,4x^{-3}\,-\,2x^{\frac{1}{2}}\,+\,6$

Differentiate: $f'(x)\,=\,\left(\frac{15}{2}\right)x^{\frac{1}{2}}\,+\,4x^{-\frac{3}{2}}\,-\,12x^{-4}\,-\,x^{-\frac{1}{2}}$

Answer: $f'(x)\,=\,(15/2)\sqrt{x}\,+\,4/x\text{^}(3/2)\,-\,12/x^4\,-\,1/\sqrt{x}$

Example 5: Engineering — Velocity and Acceleration of a Piston

A piston's displacement (cm) during one cycle is:

$$s(t) = 4t^4 - 12t^3 + 9t^2 + 2t \; (0 \le t \le 2\, s)$$

(a) Find velocity v(t) = s'(t). (b) Find acceleration a(t) = s''(t). (c) Find when acceleration is zero.

(a) $\boldsymbol{v(t)} := 16t^3 - 36t^2 + 18t + 2\, cm/s$

(b) $\boldsymbol{a(t)} := v'(t) = 48t^2 - 72t + 18\, cm/s^2$

(c) $\boldsymbol{a(t)} = \mathbf{0}$: $48t^2 - 72t + 18 = 0 \rightarrow 8t^2 - 12t + 3 = 0$

Quadratic formula: $t = (12 \pm \sqrt{(144 - 96)})/16 = (12 \pm \sqrt{48})/16 = (3 \pm \sqrt{3})/4$

$$t_1 = (3 - \sqrt{3})/4 \approx 0.317\, s \; and \; t_2 = (3 + \sqrt{3})/4 \approx 1.183\, s$$

Physical meaning: These are the times when the piston transitions from accelerating to decelerating and back — corresponding to the reversal points of the force curve.

Example 6: Engineering — Electrical Resistance and Power

The power dissipated in a resistor carrying current $i(t)$ (A) *is* $P = i^2 \cdot R$. If the current varies as $i(t) = 3t^2 - 2t + 1\, A \; and \; R = 10\, \Omega, find \frac{dP}{dt}$.

P(t) = $\boldsymbol{R \cdot i(t)^2 = 10(3t^2 - 2t + 1)^2}$:

Expand: $\boldsymbol{(3t^2 - 2t + 1)^2 = 9t^4 - 12t^3 + 10t^2 - 4t + 1}$:

$$\boldsymbol{P(t) = 90t^4 - 120t^3 + 100t^2 - 40t + 10}:$$

$$\boldsymbol{\frac{dP}{dt} = 360t^3 - 360t^2 + 200t - 40}:$$

$$\boldsymbol{At\ t = 1: \frac{dP}{dt} = 360 - 360 + 200 - 40 = 160\, W/s}.$$

Interpretation: At t = 1 s the power dissipation is increasing at 160 watts per second.

Exercises 3.1 — Basic Differentiation Rules

A. Differentiate each function:

1. $f(x) = 12x^8 - 5x^6 + 4x^4 - 2x^2 + 3x - 7$

1. $g(x) = 6x^{\frac{5}{3}} - 4x^{-2} + x^{\frac{1}{7}}$

2. $h(x) = (x^5 - 3x^3 + 2x) / x^2$ [$divide\ each\ term\ first$]

3. $k(x) = x^{\frac{1}{2}} \cdot (5x^2 - 3x + 2)$ [expand first, then differentiate]

4. $m(x) = (3x - 1/x)^2$ [expand the square]

5. $n(x) = 7x^{-\frac{4}{3}} + 12x^{\frac{8}{5}} - 15 + 3/x$

B. Applied Problems:

1. A ball is thrown vertically. Its height (m) is $h(t) = -4.9t^2 + 20t + 1.5$. Find velocity v(t) and the time at which it reaches maximum height (v = 0).

1. The cost (USD '000) to produce x tonnes of steel is $C(x) = 0.004x^3 - 0.8x^2 + 100x + 5000$. Find marginal cost C'(x) at x = 80 tonnes.

2. A thermistor has resistance $R(T) = 1200 \cdot T^{-0.85} + 30\ \Omega. Find \frac{dR}{dT}\ at\ T = 400\ K.$ Is resistance increasing or decreasing with temperature?

3. The cross-sectional area of a river at station x km downstream is $A(x) = 5x^{\frac{2}{3}} + 2x\ m^2. Find \frac{dA}{dx}\ at\ x = 8\ km.$

C. First-Principles:

1. Using the limit definition, verify $\frac{d}{dx}[x^4] = 4x^3$ by computing $\frac{[(x+h)^4 - x^4]}{h}$ and taking h→0. Show all algebraic steps.

1. From first principles, prove $\frac{d}{dx}[5x^2 - 3x] = 10x - 3$.

3.1.6 Tangent Lines and Normal Lines

The derivative f'(a) is the slope of the tangent line to the graph y = f(x) at the point (a, f(a)). Writing the equation of that tangent line is one of the most immediate and useful applications of differentiation — it underpins linear approximation, Newton's Method, and the first step of every optimisation procedure.

Tangent and Normal Line Formulas

Tangent line at (a, f(a)): y = f(a) + f'(a)·(x − a)

Normal line at (a, f(a)) — perpendicular to tangent (slope = −1/f'(a), provided f'(a) ≠ 0):

y = f(a) − (1/f'(a))·(x − a)

Mnemonic: the normal slope is the negative reciprocal of the tangent slope.

Example 3a: Tangent and Normal to a Cubic Curve

Find the tangent and normal lines to $y = x^3 - 3x$ at the point (2, 2).

Step 1 — Derivative: $y' = 3x^2 - 3$

Step 2 — Slope at x = 2: $f'(2) = 3(4) - 3 = 9$

Tangent line: $y - 2 = 9(x - 2) \rightarrow y = 9x - 16$

Normal line (slope = −1/9): $y - 2 = -(1/9)(x - 2) \rightarrow y = -x/9 + 20/9$

Numerical check (CDQ, h = 0.001): $[f(2.001) - f(1.999)] / 0.002 = [8.024 - 7.976] / 0.002 = 0.048/0.002 = 9.000$ ✓

Example 3b: Tangent to a Sine Curve — with x-Intercept

Find the equation of the tangent line to $y = \sin x$ at $x = \pi/4$. Hence find where the tangent crosses the x-axis.

At $x = \pi/4$: $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2 \approx 0.7071$; $f'(x) = \cos x \rightarrow f'(\pi/4) = \cos(\pi/4) = \sqrt{2}/2 \approx 0.7071$

Tangent line: $y - \sqrt{2}/2 = (\sqrt{2}/2)(x - \pi/4) \rightarrow y = (\sqrt{2}/2) \cdot x - (\sqrt{2}/2) \cdot \pi/4 + \sqrt{2}/2 = (\sqrt{2}/2)[x - \pi/4 + 1]$

x-intercept (y = 0): $x = \pi/4 - 1 \approx -0.2146$. The tangent line crosses the x-axis to the left of the contact point.

Geometric note: The tangent to $y = \sin x$ at $x = \pi/4$ has slope ≈ 0.707 — the same as $\sin(\pi/4)$ because sin and cos coincide at $\pi/4$. The tangent line lies below the curve for $x > \pi/4$ (since sin x is concave down for $0 < x < \pi$).

Example 3c: Engineering — Beam Deflection Slope and Tangent Line

A simply-supported beam of span L = 6 m deflects under a central point load P. The deflection curve for the left half ($0 \le x \le L/2$) is:

$$\boldsymbol{y(x) = P/(48EI) \cdot (3L^2x - 4x^3), \qquad 0 \le x \le L/2}$$

Find (a) the slope angle at x = 1 m; (b) the tangent line equation at that point. Use P = 20 kN, L = 6 m, EI = 4×10^7 N·m².

$$\boldsymbol{y'(x) = P/(48EI) \cdot (3L^2 - 12x^2) = 20000/(48 \times 4 \times 10^7) \cdot (108 - 12x^2)}$$

(a) Slope at x = 1 m: $y'(1) = [20000/(1.92 \times 10^9)] \cdot (108 - 12) = (1.042 \times 10^{-5}) \cdot 96 = 1.00 \times 10^{-3}$ rad

Slope angle $\theta = \arctan(1.00 \times 10^{-3}) \approx 0.0573°$ (very small — consistent with small-deflection beam theory).

(b) Deflection at x = 1 m: $y(1) = (1.042\times10^{-5})\cdot(108-4) = (1.042\times10^{-5})\cdot104 = 1.084\times10^{-3}$ m = 1.084 mm

Tangent line: $y - 1.084\times10^{-3} = (1.00\times10^{-3})(x-1) \rightarrow y \approx 1.00\times10^{-3}\cdot x + 8.4\times10^{-5}$ m

Engineering significance: The tangent to the deflection curve gives the rotation of the beam cross-section. A rotation of 1.00×10^{-3} rad = 0.057° is well within the Eurocode SLS limit of L/500 = 12 mm for a 6 m span — confirming the beam is structurally acceptable at this point.

Exercises 3.1.6 — Tangent and Normal Lines

A. Find the equation of the tangent and normal lines to each curve at the given point:

(i) $y = x^4 - 2x^2 + 1$ at (1, 0) (ii) $y = \sqrt{x}$ at (4, 2) (iii) $y = e^x$ at (0, 1)

(iv) $y = \ln x$ at (e, 1) (v) $y = x\cdot\cos x$ at $(\pi/2, 0)$

B. Applied:

A cam profile is described by $r(\theta) = 2 + \cos\theta$ (polar curve, plotted in Cartesian as $x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$). Find the slope dy/dx of the tangent at $\theta = \pi/3$ using parametric differentiation.

A stress-strain curve in the elastic range follows $\sigma = E\cdot\varepsilon$ where E = 200 GPa. Write the tangent line at (0.001, 200 MPa) and interpret its slope physically.

C. Proof:

Show that the tangent to $y = x^2$ at the point (a, a^2) intersects the x-axis at $x = a/2$. Hence explain why Newton's Method applied to $f(x) = x^2 - c$ gives the iteration $x_{n+1} = (x_n + c/x_n)/2$ — the classic square-root algorithm.

3.2 The Product Rule

A natural conjecture is that $[f\cdot g]' = f'\cdot g'$. Testing this on $f(x) = x^2$ *and* $g(x) = x^3$ gives $[f\cdot g]' = [x^5]' = 5x^4, but\ f'\cdot g' = 2x\cdot 3x^2 = 6x^3 \neq 5x^4$. The conjecture fails. The correct formula — the Product Rule — accounts for the interaction between both changing quantities.

3.2.1 Derivation from First Principles

Let $P(x) = f(x)\cdot g(x)$ with f, g differentiable at x. The *algebraic key* is to add and subtract a cross term $f(x)\cdot g(x+h)$ in the numerator to split it into two recognisable difference quotients:

Proof: Product Rule

$$P'(x) = lim_{\{h\to 0\}} \frac{[f(x+h)\cdot g(x+h) - f(x)\cdot g(x)]}{h}$$

$$Add\ and\ subtract\ f(x)\cdot g(x+h):$$

$$= lim \frac{\{[f(x+h)-f(x)]\cdot g(x+h) + f(x)\cdot [g(x+h)-g(x)]\}}{h}$$

$$= lim_{\{h\to 0\}} \frac{[f(x+h)-f(x)]}{h} \cdot g(x+h) + f(x)\cdot lim_{\{h\to 0\}} \frac{[g(x+h)-g(x)]}{h}$$

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

(Note: $g(x+h) \to g(x)\ as\ h \to 0$ since differentiability implies continuity)

Product Rule: $\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Mnemonic: "(d First)·Second + First·(d Second)"

Geometric Interpretation of the Product Rule

Imagine a rectangle with side lengths $u = f(x)\ and\ v = g(x)$, giving area $A = u\cdot v. When\ x\ increases\ by\ \delta x$:

1. The width increases by δu ≈ f'(x)δx, adding a vertical strip of area $\delta u\cdot v = f'(x)g(x)\delta x$.
2. The height increases by $\delta v \approx g'(x)\delta x$, adding a horizontal strip of area $u\cdot\delta v = f(x)g'(x)\delta x$.
3. The corner piece $\delta u\cdot\delta v \approx f'g'\cdot(\delta x)^2$ is negligible $as\ \delta x \to 0$.

Total rate of area change per unit x-change: $f'(x)\cdot g(x) + f(x)\cdot g'(x)$. This is the Product Rule — expressed as a rectangle area argument, it is completely intuitive.

Example 7: Product of Polynomial and Trigonometric Function

Differentiate $f(x) = (2x^3 + 5x)\cdot cos(x)$.

$$\boldsymbol{u = 2x^3 + 5x \to u' = 6x^2 + 5}:$$

$$\boldsymbol{v = cos\ x \to v' = -sin\ x}:$$

$$\boldsymbol{f'(x) = (6x^2+5)cos\ x + (2x^3+5x)(-sin\ x)}:$$

Answer: f'(x) = (6x²+5)cos x – (2x³+5x)sin x

Example 8: Product of Exponential and Power Functions

Differentiate $h(x) = x^5 \cdot e^{-2x}$.

$$\boldsymbol{u = x^5 \rightarrow u' = 5x^4;\ v = e^{-2x} \rightarrow v' = -2e^{-2x}\ (Chain\ Rule)}:$$

$$\boldsymbol{h'(x) = 5x^4 \cdot e^{-2x} + x^5 \cdot (-2e^{-2x}) = e^{-2x} \cdot x^4(5 - 2x)}:$$

Answer: $h'(x) = x^4 \cdot e^{-2x} \cdot (5 - 2x)$

Stationary points: $h'(x) = 0\ when\ x = 0\ (f(0) = 0) or\ x = \frac{5}{2}$ (positive maximum since $h(\frac{5}{2}) > 0$).

Example 9: Product Rule — Verifying the Formula for tan x

We will use the Product Rule to verify $\frac{d}{dx}[x \cdot sin\ x] at\ x = \frac{\pi}{6}$ from two approaches.

Analytical (Product Rule): $\frac{d}{dx}[x \cdot sin\ x] = sin\ x + x \cdot cos\ x$

At $\boldsymbol{x = \frac{\pi}{6}}$: $sin\left(\frac{\pi}{6}\right) + \left(\frac{\pi}{6}\right) cos\left(\frac{\pi}{6}\right) = \frac{1}{2} + (\frac{\pi}{6})(\frac{\sqrt{3}}{2}) = 0.5 + 0.4534 \approx 0.9534$

Numerical check (CDQ, h=0.001):

$$\boldsymbol{f(\frac{\pi}{6} + 0.001)}: = (\frac{\pi}{6} + 0.001) \cdot sin(\frac{\pi}{6} + 0.001) \approx 0.52428 \cdot 0.50050 \approx 0.26243$$

$$\boldsymbol{f(\frac{\pi}{6} - 0.001)}: = (\frac{\pi}{6} - 0.001) \cdot sin(\frac{\pi}{6} - 0.001) \approx 0.52228 \cdot 0.49950 \approx 0.26090$$

$$\boldsymbol{CDQ} = (\boldsymbol{0.26243 - 0.26090})/\boldsymbol{0.002}: \approx 0.9533\ \checkmark (matches\ 0.9534\ to\ 4\ d.p.)$$

3.2.2 The Extended Product Rule — Three or More Functions

For three differentiable functions u, v, w, the Product Rule generalises to:

Extended Product Rule

$$\frac{d}{dx}[u \cdot v \cdot w] = u' \cdot v \cdot w + u \cdot v' \cdot w + u \cdot v \cdot w'$$

For n factors: $\frac{d}{dx}[f_1 \cdot f_2 \cdots f_n] = \Sigma_i\, f_1 \cdot f_2 \cdots f_i' \cdots f_n$

(Differentiate exactly one factor at a time; sum all n such terms)

Example 10: Three-Factor Product

Differentiate $k(x) = x^3 \cdot e^x \cdot \sin x$.

$$\boldsymbol{u = x^3, v = e^x, w = \sin x \rightarrow u' = 3x^2, v' = e^x, w' = \cos x}:$$

$$\boldsymbol{k'(x) = 3x^2 \cdot e^x \cdot \sin x + x^3 \cdot e^x \cdot \sin x + x^3 \cdot e^x \cdot \cos x}:$$

$$= x^2 \cdot e^x \cdot (3\sin x + x \cdot \sin x + x \cdot \cos x)$$

Answer: $k'(x) = x^2 \cdot e^x \cdot [(3 + x)\sin x + x \cdot \cos x]$

Example 11: Engineering — Instantaneous Power in an AC Circuit

In an AC circuit with varying impedance, the instantaneous power is $P(t) = v(t) \cdot i(t)$, where:

$$v(t) = (50 + 10t) \cdot \sin(100\pi t)\ V \text{ and } i(t) = 4e^{-0.5t}\ A$$

Find $\frac{dP}{dt}$ $at\ t = 0$.

$$\boldsymbol{P(t) = v(t) \cdot i(t);\ apply\ Product\ Rule\ P' = v'i + vi'}:$$

$$\boldsymbol{v'(t) = 10 \cdot \sin(100\pi t) + (50 + 10t) \cdot 100\pi \cdot \cos(100\pi t)}:$$

$$\boldsymbol{i'(t) = -2e^{-0.5t}}:$$

At $t = 0$: $\boldsymbol{v(0) = 0\ (\sin 0 = 0), i(0) = 4, v'(0) = 0 + 50 \cdot 100\pi \cdot 1 = 5000\pi, i'(0) = -2}$:

$$\boldsymbol{P'(0) = v'(0) \cdot i(0) + v(0) \cdot i'(0) = 5000\pi \cdot 4 + 0 \cdot (-2) :=} \ 20000\pi \approx 62832\ W/s$$

Interpretation: At t = 0 the power is rising rapidly at ≈62.8 kW/s as the AC cycle begins.

Exercises 3.2 — The Product Rule

Differentiate using the Product Rule:

1. $f(x) = (4x^3 - x + 2) \cdot (x^2 + 3x)$
1. $g(x) = x^{\frac{2}{3}} \cdot \ln x$
2. $h(x) = (1 + x^2) \cdot \cos x$
3. $k(x) = (e^x - 1) \cdot (x^3 + 2)$
4. $m(x) = x^{-\frac{1}{2}} \cdot (x^2 - x + 1)$

5. Prove: $\frac{d}{dx}[sin^2 x] = sin(2x)$ using the Product Rule on $sin\, x \cdot sin\, x$.

Applied:

1. A spring-mass system has position $x(t) = (1 + 0.2t) \cdot cos(5t)\, m$. Find velocity dx/dt and evaluate at $t = \pi/10\ s$.

1. The bending stiffness of a tapered beam is $EI(x) = E \cdot \frac{bx^3}{12}\ where\ b = 0.1\, m, E = 70\ GPa$. The curvature is $\kappa(x) = \frac{M(x)}{EI(x)}$ where $M(x) = 500x(4 - x)$. Find $\frac{d\kappa}{dx}\ at\ x = 2\, m$.

2. Use the extended Product Rule to differentiate $f(x) = x^2 \cdot cos\, x \cdot ln\, x$.

3.3 The Quotient Rule

The Quotient Rule provides the derivative of $\frac{f(x)}{g(x)}$. It follows directly from the Product Rule — making a separate first-principles derivation unnecessary once the Product Rule is established.

3.3.1 Derivation from the Product Rule

Write $Q(x) = f(x)/g(x)\,, so\ Q(x) \cdot g(x) = f(x)$. Differentiating both sides with the Product Rule:

Proof: Quotient Rule via Product Rule

$$Q'(x) \cdot g(x) + Q(x) \cdot g'(x) = f'(x)$$

$$Q'(x) \cdot g(x) = f'(x) - Q(x) \cdot g'(x) = f'(x) - \left[\frac{f(x)}{g(x)}\right] \cdot g'(x)$$

$$= [f'(x) \cdot g(x) - f(x) \cdot g'(x)] \,/\, g(x)$$

$$Q'(x) = [f'(x) \cdot g(x) - f(x) \cdot g'(x)] \,/\, [g(x)]^2$$

Quotient Rule: $d/dx\,[\frac{f(x)}{g(x)}] = [f'(x) \cdot g(x) - f(x) \cdot g'(x)]/[g(x)]^2\ (g(x) \neq 0)$

"Derivative of the top times the bottom, minus the top times derivative of the bottom, all over bottom squared."

Common Error: Sign in the Quotient Rule

The numerator is $\boldsymbol{f'g - fg'}, not\ fg' - f'g$. The numerator is NOT symmetric — swapping the order changes the sign of the result. Always put the derivative of the **numerator** first. A helpful check: the denominator $[g(x)]^2$ is always positive ($when\ g(x) \neq 0$), so the sign of $Q'(x)$ depends solely on the numerator.

Example 12: Quotient of Polynomials

Differentiate $f(x) = (3x^2 + 1) / (x^3 - 4x)$.

$$\boldsymbol{Hi = 3x^2 + 1 \rightarrow Hi' = 6x;\ Lo = x^3 - 4x \rightarrow Lo' = 3x^2 - 4}:$$

QR: $f'(x) = [6x(x^3 - 4x) - (3x^2 + 1)(3x^2 - 4)] / (x^3 - 4x)^2$

Expand numerator: = $(6x^4 - 24x^2) - (9x^4 - 12x^2 + 3x^2 - 4) = 6x^4 - 24x^2 - 9x^4 + 9x^2 + 4$

$= -3x^4 - 15x^2 + 4$

Answer: $f'(x) = (-3x^4 - 15x^2 + 4) / (x^3 - 4x)^2$

Example 13: Deriving All Six Trigonometric Derivatives via Quotient Rule

(a) $\frac{d}{dx}[\boldsymbol{tan\ x}] = \boldsymbol{sec^2 x}$:

$$\boldsymbol{tan\ x} = \frac{\boldsymbol{sinx}}{\boldsymbol{cosx}};\ \boldsymbol{Hi' = cos\ x, Lo' = -sin\ x}: = \frac{cos^2x + sin^2x}{cos^2x} = \frac{1}{cos^2x} = sec^2x\ \checkmark$$

(b) $\frac{d}{dx}[\boldsymbol{cot\ x}] = \boldsymbol{-csc^2 x}$:

$$\boldsymbol{cot\ x} = \frac{\boldsymbol{cosx}}{\boldsymbol{sinx}};\ \boldsymbol{Hi' = -sin\ x, Lo' = cos\ x}: = \frac{-sin^2x - cos^2x}{sin^2x} = -\frac{1}{sin^2x} = -csc^2x\ \checkmark$$

(c) $\frac{d}{dx}[\boldsymbol{sec\ x}] = \boldsymbol{sec\ x\ tan\ x}$:

$$\boldsymbol{sec\ x} = \frac{\boldsymbol{1}}{\boldsymbol{cosx}};\ \boldsymbol{Hi' = 0, Lo' = -sin\ x}: = sin\frac{x}{cos^2x} = \left(\frac{1}{cos}x\right)\left(sin\frac{x}{cos}x\right)$$
$$= sec\ x \cdot tan\ x\ \checkmark$$

(d) $\frac{d}{dx}[\boldsymbol{csc\ x}] = \boldsymbol{-csc\ x\ cot\ x}$:

$$\boldsymbol{csc\ x} = \frac{\boldsymbol{1}}{\boldsymbol{sinx}};\ \boldsymbol{Hi' = 0, Lo' = cos\ x}: = -\frac{cosx}{sin^2x} = -\left(\frac{1}{sin}x\right)\left(\frac{cosx}{sinx}\right) = -csc\ x \cdot cot\ x\ \checkmark$$

Trig Function	Quotient Form	Derivative
$\boldsymbol{sin\ x}$	$sin\frac{x}{1}$	$cos\ x$

$\boldsymbol{\cos x}$	$\cos\frac{x}{1}$	$-\sin x$
$\boldsymbol{\tan x}$	$\frac{\sin x}{\cos x}$	$\sec^2 x$
$\boldsymbol{\cot x}$	$\frac{\cos x}{\sin x}$	$-\csc^2 x$
$\boldsymbol{\sec x}$	$\frac{1}{\cos x}$	$\sec x \cdot \tan x$
$\boldsymbol{\csc x}$	$\frac{1}{\sin x}$	$-\csc x \cdot \cot x$

Example 14: Engineering — Voltage Divider with Frequency-Dependent Impedance

In an RC voltage divider, the gain (ratio of output to input voltage) as a function of angular frequency ω is:

$$G(\omega) = 1 / \sqrt{(1 + \omega^2 R^2 C^2)} = (1 + \omega^2 R^2 C^2)^{-\frac{1}{2}}$$

With $R = 1000\,\Omega, C = 1\,\mu F$ $(RC = 10^{-3} s)$, $find\ \frac{dG}{d\omega}$ and evaluate at the cut-off frequency $\omega_c = 1/(RC) = 1000\ rad/s$.

Let $\boldsymbol{u = 1 + \omega^2 R^2 C^2 \rightarrow G = u^{-\frac{1}{2}}}$:

$$\boldsymbol{\frac{dG}{d\omega} = -\left(\frac{1}{2}\right) u^{-\frac{3}{2}} \cdot \frac{du}{d\omega}}\ (\textbf{Chain Rule}):$$

$$\boldsymbol{\frac{du}{d\omega} = 2\omega R^2 C^2}:$$

$$\boldsymbol{\frac{dG}{d\omega} = -\omega R^2 C^2 \cdot (1 + \omega^2 R^2 C^2)^{-\frac{3}{2}}}:$$

$$\textbf{At } \boldsymbol{\omega = 1000}\ \left(\boldsymbol{\omega = \frac{1}{RC}}\right): \boldsymbol{\omega^2 R^2 C^2 = 1 \rightarrow 1 + \omega^2 R^2 C^2 = 2}:$$

$$\boldsymbol{\frac{dG}{d\omega} = -(1/RC) \cdot R^2 C^2 \cdot 2^{-\frac{3}{2}} = -RC/2^{\frac{3}{2}} = -10^{-3}/2\sqrt{2} \approx -3.54 \times 10^{-4} \left(\frac{rad}{s}\right)^{-1}}\ :$$

Interpretation: At the cut-off frequency, the gain is decreasing at $3.54 \times 10^{-4}\ per\ rad/s$ — this rate characterises the roll-off of the low-pass filter.

Exercises 3.3 — The Quotient Rule

Differentiate:

1. $f(x) = \frac{x^4 - 3x + 1}{2x^2 + 5}$
1. $g(x) = \frac{cosx}{x^2 + 4}$
2. $h(x) = \frac{(e^x + e^{-x})}{2}$ [this is cosh x; find its derivative]
3. $k(x) = \frac{2\sqrt{x} - 1}{\sqrt{x} + 3}$
4. Using the Quotient Rule, prove $\frac{d}{dx}[cot\ x] = -csc^2 x$.
5. Prove $\frac{d}{dx}[csc\ x] = -csc\ x \cdot cot\ x$.

Applied:

1. The Biot number for heat transfer is $Bi(h) = h \cdot \frac{L}{k}$ where h is the convective coefficient $(W/m^2K), L = 0.05\ m, k = 50\ W/mK$. If h varies with flow velocity v as $h(v) = 30v^{0.7}$, find $\frac{dBi}{dv}\ at\ v = 5\ m/s$.
1. The Michaelis-Menten enzyme kinetics model gives reaction rate $r(S) = V_{\max} \cdot \frac{S}{K_m + S}$. Find dr/dS and show it is always positive (rate always increases with substrate concentration S).
2. An impedance $Z(f) = R\ /\ (1 + j2\pi fRC)$ has magnitude $|Z(f)| = R/\sqrt{(1 + (2\pi fRC)^2)}. Find\ d|Z|/df$.

3.4 The Chain Rule

The Chain Rule handles composite functions — functions of the form y = f(g(x)) where one function acts on the output of another. It is the most frequently needed rule in engineering applications, because nearly every physical quantity involves composition: temperature depends on position which depends on time; pressure depends on density which depends on volume which depends on temperature.

3.4.1 Intuitive Motivation via Rate Chains

Suppose u depends on x and y depends on u. If u increases at 3 units per unit of x ($\frac{du}{dx} = 3$), and y increases at 5 units per unit of u ($\frac{dy}{du} = 5$), then intuitively y increases at 15 units per unit of x:

The Chain of Rates

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5 \cdot 3 = 15$$

This rate-multiplication idea is the essence of the Chain Rule. Leibniz notation makes it look like cancellation of fractions — though it is not fraction cancellation but rather a deep property of limits.

3.4.2 Formal Derivation

Let $y = f(u), u = g(x)$, both differentiable. Set $\Delta u = g(x+h) - g(x), \Delta y = f(u + \Delta u) - f(u)$.

Proof: Chain Rule

When $\Delta u \neq 0$:

$$\frac{\Delta y}{h} = \left[\frac{\Delta y}{\Delta u}\right] \cdot \left[\frac{\Delta u}{h}\right] = \frac{[f(u+\Delta u) - f(u)]}{\Delta u} \cdot \frac{[g(x+h) - g(x)]}{h}$$

$$As\ h \to 0: \Delta u \to 0\ (g\ continuous), so\ [\frac{\Delta y}{\Delta u}] \to f'(u)\ and\ [\frac{\Delta u}{h}] \to g'(x)$$

Therefore: $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

(The case $\Delta u = 0$ for small h is handled by defining $F(\Delta u) = \frac{[f(u+\Delta u) - f(u)]}{\Delta u}$ continuously at $\Delta u = 0$, giving $F(0) = f'(u)$; then $\Delta y = F(\Delta u) \cdot \Delta u\ always.$)

Chain Rule (function notation): $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Chain Rule (Leibniz notation): $\frac{dy}{dx} = (\frac{dy}{du}) \cdot (\frac{du}{dx})$

Procedure:

1. Identify outer function f and inner function $u = g(x)$.
2. Differentiate outer: $f'(u)$ — write the answer in terms of $u = g(x)$.
3. Multiply by the derivative of the inner: $g'(x)$.

Example 15: Chain Rule — Power of a Composite

Differentiate $y = (7x^3 - 4x + 2)^6$.

Outer: $\boldsymbol{u^6 \to 6u^5}$; ***Inner***: $\boldsymbol{u = 7x^3 - 4x + 2 \to u' = 21x^2 - 4}$:

$$\boldsymbol{y' = 6(7x^3 - 4x + 2)^5 \cdot (21x^2 - 4)}:$$

Answer: $y' = 6(21x^2 - 4)(7x^3 - 4x + 2)^5$

Example 16: Chain Rule — Exponential Composite

Differentiate $y = e^{4x^2-3x+1}$.

Outer: $\boldsymbol{e^u \to e^u}$; ***Inner***: $\boldsymbol{u = 4x^2 - 3x + 1 \to u' = 8x - 3}$:

$$\boldsymbol{y' = e^{4x^2-3x+1} \cdot (8x - 3)}:$$

Answer: $y' = (8x - 3) \cdot e^{4x^2-3x+1}$

Example 17: Chain Rule — Trigonometric Composite

Differentiate $y = sin^3(2x) = [sin(2x)]^3$.

Outer: $\boldsymbol{u^3 \to 3u^2}$; $\boldsymbol{Inner_1}$: $\boldsymbol{sin(v) \to cos(v)}$; $\boldsymbol{Inner_2}$: $\boldsymbol{v = 2x \to 2}$:

$$\boldsymbol{y' = 3sin^2(2x) \cdot cos(2x) \cdot 2 = 6sin^2(2x) \cdot cos(2x)}:$$

Answer: $y' = 6sin^2(2x) \cdot cos(2x) = 3sin(2x) \cdot \frac{sin(4x)}{2}$

Using double-angle identities: $6sin^2(2x)cos(2x) = 3sin(2x) \cdot \frac{[2sin(2x)cos(2x)]}{sin(2x)} = 3sin^2(2x)$... The simplified form 3sin²(4x)/2 may also be written using further identities.

Example 18: Three-Layer Chain Rule

Differentiate $y = e^{sin(x^2)}$.

Layer 1 (outermost): $\boldsymbol{e^u \to e^u, u = sin(x^2)}$**:**

Layer 2: $\boldsymbol{sin(v) \to cos(v), v = x^2}$:

Layer 3: $\mathbf{x^2 \to 2x}$:

Chain Rule: $\boldsymbol{y' = e^u \cdot cos(v) \cdot 2x = e\text{^}(sin(x^2)) \cdot cos(x^2) \cdot 2x}$:

Answer: $y' = 2x \cdot cos(x^2) \cdot e^{sin(x^2)}$

Example 19: Chain Rule + Product Rule

Differentiate $f(x) = x^4 \cdot e^{sin\,x}$.

Product Rule: $\boldsymbol{u = x^4 \to 4x^3}$; $\boldsymbol{v = e^{sin\,x} \to e^{sin\,x} \cdot cos\,x}$ $(\boldsymbol{Chain\ Rule})$:

$$\boldsymbol{f'(x) = 4x^3 \cdot e^{sin\,x} + x^4 \cdot cos\,x \cdot e^{sin\,x}}:$$

$$= x^3 \cdot e^{\sin x} \cdot (4 + x \cdot \cos x)$$

Answer: $f'(x) = x^3 \cdot e^{\sin x} \cdot (4 + x \cdot \cos x)$

Example 20: Engineering — Damped Oscillation Velocity

A damped mechanical oscillator has displacement:

$$y(t) = 0.3 \cdot e^{-1.5t} \cdot \cos(8t + 0.4)\ m$$

Find $v(t) = \frac{dy}{dt}$ *and the velocity at* $t = 0$.

Product Rule: $\boldsymbol{u = 0.3e^{-1.5t} \rightarrow u' = -0.45e^{-1.5t}}$;

v= $\boldsymbol{\cos(8t + 0.4) \rightarrow v' = -8\sin(8t + 0.4)}$:

$$\frac{dy}{dt} = -0.45e^{-1.5t}\cos(8t + 0.4) + 0.3e^{-1.5t} \cdot (-8\sin(8t + 0.4)):$$

$$= e^{-1.5t}[-0.45\cos(8t + 0.4) - 2.4\sin(8t + 0.4)]$$

At $\boldsymbol{t = 0}$: $\boldsymbol{e^0 = 1, \cos(0.4) \approx 0.9211, \sin(0.4) \approx 0.3894}$:

$$v(0) = -0.45(0.9211) - 2.4(0.3894) = -0.4145 - 0.9346 \approx -1.349\ m/s:$$

Physical meaning: At t = 0 the oscillator starts at $y(0) = 0.3\cos(0.4) \approx 0.276\ m$ and moves toward equilibrium at 1.35 m/s.

Example 21: Engineering — Arrhenius Reaction Rate

The rate constant for a chemical reaction follows the Arrhenius equation:

$$k(T) = A \cdot e^{-\frac{E_a}{RT}}$$

Find dk/dT. With $E_a = 80\ kJ/mol, R = 8.314\ J/(mol \cdot K)$, *evaluate at* $T = 500\ K$.

$$\textbf{Inner } u = -\frac{E_a}{RT} = -E_a \cdot R^{-1} \cdot T^{-1} \rightarrow \frac{du}{dT} = \frac{E_a}{RT^2}:$$

$$\frac{dk}{dT} = A \cdot e^{-\frac{E_a}{RT}} \cdot \frac{E_a}{RT^2} = k(T) \cdot \frac{E_a}{RT^2}:$$

$$\text{At } T = 500: \frac{dk}{k} = E_a/(RT^2) = 80000/(8.314 \times 250000) = 0.0385\ K^{-1}:$$

Interpretation: A 1 K temperature rise at 500 K increases the rate constant by 3.85%. This high sensitivity is why precise reactor temperature control is critical in the chemical industry.

Exercises 3.4 — The Chain Rule

Differentiate each composite function:

1. $f(x) = (x^6 - 2x^3 + 5)^{12}$
1. $g(x) = \sqrt{(5x^4 - 3x + 1)}$
2. $h(x) = e^{2x^3 - x^2}$
3. $k(x) = sin(cos\, x)$ [note: this is a composite]
4. $m(x) = cos^2(x^3) = [cos(x^3)]^2$
5. $n(x) = ln(x^2 + e^x)$
6. $p(x) = [tan(2x + 1)]^{\frac{1}{3}}$
7. $q(x) = sin(e^{cos\, x})$ [identify three layers]

Chain + Product or Quotient:

1. $r(x) = x^2 \cdot sin(x^3 + 1)$
1. $s(x) = \frac{e^{x^2}}{x^2 + 1}$
2. $u(x) = \frac{cos(ln\, x)}{x}$

Applied:

1. Temperature in an insulated rod: $T(x) = T_0 \cdot e^{-\beta x^2}, \beta = 0.08$ m⁻². Find dT/dx and the temperature gradient at x = 2 m.
1. In a PID controller, the derivative term involves de/dt where $e(t) = r(t) - y(t). If\ r(t) = 5sin(2t)\ and\ y(t) = 4e^{-\frac{t}{3}}, find\ de/dt.$
2. Derive $\frac{d}{dx}[arcsin\, x] = 1/\sqrt{(1 - x^2)}$ using the Chain Rule on x = sin y.

3.5 First-Principles Verification of Differentiation Rules

Verification of analytical derivatives against numerical approximations is essential in engineering practice. Errors in differentiation — even experienced practitioners make sign errors or miss a chain rule factor — can propagate into incorrect design decisions. The central-difference quotient provides a rapid and reliable check.

3.5.1 The Central-Difference Quotient

Numerical Differentiation — Central-Difference Formula

Central-Difference Quotient (CDQ): $f'(x_0) \approx [f(x_0+h) - f(x_0-h)] / (2h)$

Error: $CDQ - f'(x_0) = O(h^2) \rightarrow$ halving h reduces error by factor ~4

Compare: one-sided difference: $[f(x_0+h) - f(x_0)]/h \; has \; error \; O(h) \; only$

The O(h²) convergence of the CDQ means that a step size of h = 0.001 typically gives 6 significant figures of accuracy, sufficient to confirm most analytical results. For h = 0.01 we expect 4 significant figures, and for h = 0.1 roughly 2–3.

Example 22: Verify $d/dx[x^5] = 5x^4$ at $x = 2$

Exact: $5 \cdot 2^4 = 80$

h	$CDQ = \frac{[f(2+h) - f(2-h)]}{2h}$
0.5	$\frac{[(2.5)^5 - (1.5)^5]}{1.0} = \frac{97.656 - 7.594}{1.0} = 90.062$
0.1	$\frac{[(2.1)^5 - (1.9)^5]}{0.2} = \frac{40.841 - 24.761}{0.2} = 80.40$
0.01	$\frac{[(2.01)^5 - (1.99)^5]}{0.02} \approx 80.004$
0.001	≈ 80.000004
$\rightarrow 0$	$\rightarrow 80$ ✓

Error ratio: Halving h from 0.1 to 0.05: error goes from 0.400 to ~0.100 (factor 4) — confirming O(h²).

Example 23: Verify the Product Rule: f$(x) = x^2 \cdot cos\, x$ at $x = \pi/3$

Product Rule: $f'(x) = 2x \cdot cos\, x - x^2 \cdot sin\, x$

$$At \; x = \frac{\pi}{3}: f'(\frac{\pi}{3}) = 2(\frac{\pi}{3})cos(\frac{\pi}{3}) - (\frac{\pi}{3})^2 sin(\frac{\pi}{3}) = (\frac{2\pi}{3})(\frac{1}{2}) - (\frac{\pi^2}{9})(\frac{\sqrt{3}}{2})$$

$$= \frac{\pi}{3} - \pi^2\sqrt{3}/18 \approx 1.0472 - 0.9514 \approx 0.0958$$

$$CDQ \; at \; h = 0.001:$$

$$f(\frac{\pi}{3} + 0.001) = (\frac{\pi}{3} + 0.001)^2 \cdot cos(\frac{\pi}{3} + 0.001) \approx 1.10035^2 \cdot 0.49950 \approx 0.60447:$$

$f(\frac{\pi}{3} - 0.001) = (\frac{\pi}{3} - 0.001)^2 \cdot cos(\frac{\pi}{3} - 0.001) \approx 1.09835^2 \cdot 0.50050 \approx 0.60256$:

$CDQ = (0.60447 - 0.60256)/0.002 \approx 0.0955$ ($agrees\ with\ 0.0958\ to\ 3\ d.p.$) ✓:

Example 24: Verify the Chain Rule: $y = e^{x^2}$ at $x = 1$

Chain Rule: $y' = 2x \cdot e^{x^2} \rightarrow y'(1) = 2e \approx 5.4366$

h	$CDQ = \frac{[e^{(1+h)^2} - e^{(1-h)^2}]}{2h}$
0.1	$\frac{[e^{1.21} - e^{0.81}]}{0.2} = \frac{[3.3535 - 2.2479]}{0.2} \approx 5.528$
0.01	$\frac{[e^{1.0201} - e^{0.9801}]}{0.02} \approx 5.4421$
0.001	≈ 5.4370
$\rightarrow 0$	$\rightarrow 2e \approx 5.4366$ ✓

3.5.2 Master Summary Table — All Differentiation Rules

Rule	Formula	Key Requirement
Constant	$d/dx[c] = 0$	c is constant
Power	$d/dx[x^n] = n \cdot x^{n-1}$	$n \in \mathbb{R}$
Scalar Multiple	$d/dx[kf] = k \cdot f'$	k is constant
Sum	$d/dx[f + g] = f' + g'$	f, g differentiable
Difference	$d/dx[f - g] = f' - g'$	f, g differentiable
Product	$d/dx[fg] = f'g + fg'$	f, g differentiable
Quotient	$d/dx[f/g] = (f'g - fg')/g^2$	$g \neq 0$
Chain	$d/dx[f(g)] = f'(g) \cdot g'$	both differentiable
Extended Product	$d/dx[uvw] = u'vw + uv'w + uvw'$	u, v, w differentiable
sin x	$cos\ x$	—
cos x	$-sin\ x$	—
tan x	sec^2x	$x \neq \pi/2 + n\pi$
cot x	$-csc^2x$	$x \neq n\pi$

$\boldsymbol{\sec x}$	$\sec x \cdot \tan x$	$x \neq \pi/2 + n\pi$
$\boldsymbol{\csc x}$	$-\csc x \cdot \cot x$	$x \neq n\pi$
$\boldsymbol{e^x}$	e^x	—
$\boldsymbol{a^x}$	$a^x \cdot \ln a$	$a > 0, a \neq 1$
$\boldsymbol{\ln x}$	$1/x$	$x > 0$
$\boldsymbol{\log_a x}$	$1/(x \cdot \ln a)$	$x > 0, a > 0, a \neq 1$

3.6 Higher-Order Derivatives

The derivative f'(x) is itself a function that can be differentiated again. The result is the second derivative f''(x), and the process continues to any order. Higher-order derivatives encode physical information beyond simple rates of change: the second derivative is acceleration and curvature; the third is jerk and rate of change of curvature; the fourth appears in beam deflection theory.

3.6.1 Notation and Physical Interpretations

Notation	Physical/Engineering Meaning
$f'(x) = dy/dx$	$Velocity;\ slope\ of\ tangent;\ rate\ of\ change$
$f''(x) = d^2y/dx^2$	$Acceleration;\ curvature\ \kappa \approx f''/(1+f'^2)\text{^}(3/2)$
$f'''(x) = d^3y/dx^3$	$Jerk\ (rate\ of\ change\ of\ acceleration);\ torsion$
$f^{(4)}(x) = d^4y/dx^4$	$Snap/jounce;\ EI \cdot y^{(4)} = w(x)\ in\ beam\ theory$
$f^{(n)}(x) = d^ny/dx^n$	$nth\ derivative —\ Taylor\ series\ coefficient\ \times n!$

Example 25: Computing All Derivatives — Polynomial

$$Find\ all\ non-zero\ derivatives\ of\ f(x) = 3x^6 - 5x^4 + x^3 - 2x + 7.$$

$$\boldsymbol{f'(x)} = 18x^5 - 20x^3 + 3x^2 - 2$$

$$\boldsymbol{f''(x)} = 90x^4 - 60x^2 + 6x$$

$$\boldsymbol{f'''(x)} = 360x^3 - 120x + 6$$

$$\boldsymbol{f^{(4)}(x)} = 1080x^2 - 120$$

$$\boldsymbol{f^{(5)}(x)} = 2160x$$

$$\boldsymbol{f^{(6)}(x)} = 2160\ (constant)$$

$$f^{(7)}(x) = 0 \ (and\ all\ higher\ orders\ are\ zero)$$

Example 26: Harmonic Oscillator Verification

Show that y = $A \cdot sin(\omega t + \varphi)\ satisfies\ y'' + \omega^2 y = 0$.

$$y' = A\omega \cdot cos(\omega t + \varphi):$$

$$y'' = -A\omega^2 \cdot sin(\omega t + \varphi) = -\omega^2 y:$$

$$y'' + \omega^2 y = -\omega^2 y + \omega^2 y = 0\ \checkmark:$$

Engineering significance: This ODE governs undamped vibrations — any mass-spring system, LC circuit, or pendulum (for small angles) satisfies it. The general solution y = A·sin(ωt+φ) is confirmed by the calculation above.

Example 27: Second Derivative — Curvature of a Beam

For a simply-supported beam carrying distributed load $w_0 = 15\ kN/m, span\ L = 10\ m$, the deflection is:

$$y(x) = (w_0/24EI) \cdot x^2 \cdot (L - x)^2$$

Find $y'(x), y''(x)$, and the bending moment $M(x) = EI \cdot y''$.

$$y(x) = (w_0/24EI) \cdot (L^2x^2 - 2Lx^3 + x^4):$$

$$y'(x) = (w_0/24EI) \cdot (2L^2x - 6Lx^2 + 4x^3):$$

$$y''(x) = (w_0/24EI) \cdot (2L^2 - 12Lx + 12x^2) = (w_0/12EI) \cdot (L^2 - 6Lx + 6x^2):$$

$$M(x) = EI \cdot y'' = (w_0/12) \cdot (L^2 - 6Lx + 6x^2):$$

$$Max\ M\ at\ x = L/2: M(L/2) = (w_0/12) \cdot (L^2 - 3L^2 + 3L^2/2) = w_0L^2/24:$$

Numerically: $M_{max} = 15 \times 100/24 = 62.5$ kN·m (classical result ✓):

3.6.2 General nth Derivative Formulas

$f(x)$	$f^{(n)}(x)$
$x^m\ (m \geq n)$	$m!/(m-n)! \cdot x^{m-n}$
e^x	e^x
e^{ax}	$a^n \cdot e^{ax}$
$sin(ax)$	$a^n \cdot sin(ax + n\pi/2)$

$cos(ax)$	$a^n \cdot cos(ax + n\pi/2)$
$ln\, x$	$(-1)^{n+1} \cdot (n-1)!\,/\,x^n\ (n \geq 1)$
$1/(x+a)$	$(-1)^n \cdot n!/(x+a)^{n+1}$

Exercises 3.6 — Higher-Order Derivatives

Find the indicated derivative:

1. $f(x) = 5x^8 - 4x^5 + 2x^2;\ find\ f^{(4)}(x)$
1. $g(x) = e^x \cdot sin\, x;\ find\ g''(x)\ and\ verify\ g'' - 2g' + 2g = 0$
2. $h(x) = ln\, x;\ derive\ a\ general\ formula\ for\ h^{(n)}(x), n \geq 1$
3. $y = cos(3x) \cdot e^{-x};\ find\ y''\ and\ verify\ y'' + 2y' + 10y = 0$

Applied:

1. A vehicle's position is $s(t) = t^4/12 - 2t^3 + 9t^2 - 12t\ m. Find\ v(t), a(t), and\ j(t) = s'''(t)\ (jerk). At\ what\ time\ is\ a(t) = 0?$
1. The temperature distribution in a fin is $T(x) = T_\infty + T_0 - T_\infty) \cdot cosh(m(L - x))/cosh(mL)$.

   Find $dT/dx\ and\ d^2T/dx^2. Show\ d^2T/dx^2 = m^2(T - T_\infty).$

3.7 Implicit Differentiation

An equation $F(x, y) = 0$ defines y *implicitly* as a function of x — we do not have an explicit formula y = f(x), but the equation still constrains y for each x. Implicit differentiation uses the Chain Rule to differentiate both sides of such an equation with respect to x, then solves algebraically for dy/dx.

Implicit Differentiation — Key Principle

Key principle: When differentiating a function of y with respect to x,

multiply by dy/dx (Chain Rule, since y depends on x):

$$\frac{d}{dx[y^2]} = 2y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx[sin\, y]} = cos\, y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx[e^y]} = e^y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx[f(y)]} = f'(y) \cdot \frac{dy}{dx}$$

Example 28: Circle — Implicit Differentiation

Find $dy/dx\ for\ x^2 + y^2 = R^2$.

Differentiate both sides w.r.t. x: $2x + 2y \cdot (dy/dx) = 0$

Solve: $dy/dx = -x/y$

Check: At (R, 0) the denominator is zero — there is a vertical tangent. At (R/√2, R/√2) the slope is −1. These are geometrically expected results for a circle.

Example 29: Implicit Differentiation with Product Term

Find $dy/dx\ for\ x^2y^3 + 2x = 5y + 3$.

Differentiate each term:

$\frac{d}{dx}[x^2y^3]$: ***Product Rule* + *Chain Rule***: $= 2xy^3 + x^2 \cdot 3y^2 \cdot (\frac{dy}{dx}) = 2xy^3 + 3x^2y^2 \cdot y'$

$$\frac{d}{dx}[2x] = 2; \frac{d}{dx[5y]} = 5y'; \frac{d}{dx}[3] = 0:$$

Full equation: $2xy^3 + 3x^2y^2 \cdot y' + 2 = 5y'$

Group y' terms: $y'(3x^2y^2 - 5) = -2xy^3 - 2$

Solve: $y' = -\frac{2xy^3+2}{(3x^2y^2-5)} = -\frac{2(xy^3+1)}{3x^2y^2-5}$

Example 30: Implicit Second Derivative — Ellipse

Find $dy/dx\ and\ d^2y/dx^2\ for\ x^2/9 + y^2/4 = 1$.

Differentiate: $2x/9 + (2y/4)y' = 0 \rightarrow y' = -4x/(9y)$

Differentiate again (Quotient Rule): $y'' = -4[y - x \cdot y'] / (9y^2)$

Substitute y': $y'' = -4[y + 4x^2/(9y)] / (9y^2) = -4[9y^2 + 4x^2] / (81y^3)$

Using ellipse equation $9y^2 + 4x^2 = 36$ (*multiply* $x^2/9 + y^2/4 = 1$ *by* 36):

$$\boldsymbol{y'' = -4(36)/(81y^3) = -16/(9y^3)}:$$

Observation: $y'' < 0 \; for \; y > 0$ (concave down on the upper ellipse), $y'' > 0 \; for \; y < 0$ (concave up on the lower ellipse). This matches the geometry perfectly.

Example 31: Engineering — van der Waals Gas

The van der Waals equation $(P + a/V^2)(V - b) = RT$ relates pressure P, volume V, temperature T (constant), and gas-specific constants a, b. Find dP/dV.

Expand: $\boldsymbol{PV - Pb + a/V - ab/V^2 = RT}$**:**

Differentiate w.r.t. V (P is implicit function of V):

$(\frac{dP}{dV})V + P + (-\frac{a}{V^2}) \cdot (\frac{dP}{dV})$**·… — wait, differentiate correctly:**

$$\frac{d}{dV}[(P + \frac{a}{V^2})(V - b)] = 0: (\frac{dP}{dV} - \frac{2a}{V^3})(V - b) + (P + \frac{a}{V^2}) = 0$$

Solve for $\frac{dP}{dV}$: $(\frac{dP}{dV})(V - b) = -(P + a/V^2) + 2a(V - b)/V^3$

$$\frac{dP}{dV} = [-\left(P + \frac{a}{V^2}\right) + \frac{2a(V - b)}{V^3}] \; / \; (V - b):$$

Critical point: Setting $dP/dV = 0 \; and \; d^2P/dV^2 = 0$ simultaneously locates the critical point of the phase diagram — fundamental in thermodynamic analysis of real gases.

Exercises 3.7 — Implicit Differentiation

Find dy/dx:

1. $x^4 + y^4 = 16$
1. $x^2y + xy^2 = 6$
2. $sin\, x + cos\, y = 1$
3. $e^{xy} = 2x + y$
4. $x \cdot ln\, y + y \cdot ln\, x = 4$

Find d²y/dx²:

1. $x^2 - 3y^2 = 9 \; (hyperbola)$
1. $xy = 4 \; (rectangular \; hyperbola)$

Applied:

1. The ideal diode equation is $I = I_0(e^{\frac{V}{V_t}} - 1)$. Treating V as a function of I, find dV/dI (the differential resistance).

1. A pressure vessel satisfies $\frac{\sigma_h}{r} + \frac{\sigma_a}{t} = \frac{p}{t}$ where r is radius, t is thickness, and σ_h, σ_a are hoop and axial stresses. If $\sigma_a = \sigma_h \cdot \frac{r}{2t}$ (closed ends), find $d\sigma_h/dr$.

3.8 Logarithmic Differentiation

When a function has a *variable exponent* — such as x^x or $(\sin x)^{\cos x}$ — neither the Power Rule (which requires a constant exponent) nor the Exponential Rule (which requires a constant base) applies directly. The solution is to take the natural logarithm of both sides, simplify using logarithm laws, differentiate implicitly, and solve for y'. This technique is also powerful for products and quotients of many factors, where it avoids repeated application of the Product and Quotient Rules.

3.8.1 Logarithm Laws Used in This Technique

Essential Logarithm Laws and Derivative

$$ln(AB) = ln\,A + ln\,B\ (product \rightarrow sum)$$

$$ln(A/B) = ln\,A - ln\,B\ (quotient \rightarrow difference)$$

$$ln(A^n) = n \cdot ln\,A\ (power \rightarrow multiplier)$$

$$d/dx[ln\,y] = (1/y) \cdot (dy/dx)\ (Chain\ Rule\ on\ ln\,y)$$

Example 32: Differentiating $y = x^x$

Differentiate $y = x^x\ (x > 0)$.

Take ln: $\boldsymbol{ln\,y = x \cdot ln\,x}$:

Differentiate: $\boldsymbol{(\frac{1}{y})y' = ln\,x + x \cdot (\frac{1}{x}) = ln\,x + 1}$:

Solve: $\boldsymbol{y' = y(1 + ln\,x) = x^x(1 + ln\,x)}$:

Answer: $d/dx[x^x] = x^x(1 + ln\,x)$

Minimum of x^x: $y' = 0\ when\ 1 + ln\,x = 0 \rightarrow x = e^{-1} = 1/e. The\ minimum\ value\ is\ \left(\frac{1}{e}\right)^{\frac{1}{e}}$.

Example 33: Differentiating $y = (\sin x)^{\tan x}$

Differentiate $y = (\sin x)^{\tan x}$.

$$\ln y = \tan x \cdot \ln(\sin x):$$

$$\left(\frac{1}{y}\right)y' = \sec^2 x \cdot \ln(\sin x) + \tan x \cdot (\cos x/\sin x):$$

$$= \sec^2 x \cdot \ln(\sin x) + \cos x/\sin x \cdot \sin x/\cos x$$

$$= \sec^2 x \cdot \ln(\sin x) + 1$$

$$y' = (\sin x)^{\tan x} \cdot [1 + \sec^2 x \cdot \ln(\sin x)]:$$

Answer: $y' = (\sin x)^{\tan x} \cdot [1 + \sec^2 x \cdot \ln(\sin x)]$

Example 34: Log Differentiation — Complex Product

Differentiate $y = [(x^2+1)^4 \cdot (3x-2)^3] / [(x+5)^2 \cdot (x^2+4x)^{\frac{1}{2}}]$.

$$\ln y = 4\ln(x^2+1) + 3\ln(3x-2) - 2\ln(x+5) - (1/2)\ln(x^2+4x):$$

$$\left(\frac{1}{y}\right)y' = 8x/(x^2+1) + 9/(3x-2) - 2/(x+5) - (x+2)/(x^2+4x):$$

$$y' = y \cdot [8x/(x^2+1) + 9/(3x-2) - 2/(x+5) - (x+2)/(x^2+4x)]:$$

where y is the original expression. This single formula replaces what would be several nested applications of the Product and Quotient Rules.

Example 35: Deriving $d/dx[a^x]$ and $d/dx[\log_a x]$

(a) $d/dx[a^x]$:

$$y = a^x \rightarrow \ln y = x \cdot \ln a \rightarrow (1/y)y' = \ln a \rightarrow y' = a^x \cdot \ln a:$$

$$d/dx[a^x] = a^x \cdot \ln a \ (a > 0, a \neq 1)$$

(b) $d/dx[\log_a x]$:

$$y = \log_a x \rightarrow a^y = x \rightarrow \text{differentiate implicitly}:$$

$$a^y \cdot \ln a \cdot y' = 1 \rightarrow y' = 1/(a^y \cdot \ln a) = 1/(x \cdot \ln a):$$

$$d/dx[\log_a x] = 1/(x \cdot \ln a) \ (x > 0, a > 0, a \neq 1)$$

Special cases: $a = e$ *gives* $\frac{d}{dx}[\ln x] = \frac{1}{x}$; $a = 10$ *gives* $\frac{d}{dx}[\log^{10} x] = \frac{1}{x \cdot \ln 10} \approx \frac{0.4343}{x}$.

Exercises 3.8 — Logarithmic Differentiation

Use logarithmic differentiation:

1. $y = x^{\cos x}$
1. $y = (\tan x)^x$
2. $y = \frac{\left[(x+1)^2(x^2+3)^4\right]}{[(2x-1)^3(x+4)]}$
3. $y = \left(1 + \frac{3}{x}\right)^x$ $[find\ the\ limit\ of\ this\ as\ x \to \infty]$
4. $y = x^{x^x}$ $[iterated\ exponential — differentiate\ carefully]$

Derivations:

1. Using logarithmic differentiation, prove the Power Rule $\frac{d}{dx}[x^n] = nx^{n-1}$ for all real n.
1. Derive $\frac{d}{dx}[\arctan x] = 1/(1+x^2)$ using implicit differentiation on x = tan y.

3.9 Parametric Differentiation

Many engineering curves — gear tooth profiles, cycloids (piston paths in rotary engines), satellite orbits, robot arm trajectories — are most naturally described by **parametric equations**: $x = x(t)\ and\ y = y(t)$, where t is a parameter (often time, but not necessarily). The Chain Rule gives us dy/dx in terms of derivatives with respect to the parameter t.

Parametric Differentiation Formulas

If $x = x(t)\ and\ y = y(t)$ are both differentiable, then:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\ (provided \frac{dx}{dt} \neq 0)$$

Second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dt}\frac{\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

Derivation: By the Chain Rule, $\frac{dy}{dt} = \left(\frac{dy}{dx}\right) \cdot \left(\frac{dx}{dt}\right)$. Dividing both sides by dx/dt (assuming $\frac{dx}{dt} \neq 0$) $gives\ dy/dx = (\frac{dy}{dt})/(\frac{dx}{dt})$. This is essentially the Chain Rule written in reverse.

Example 36: Cycloid — Path of a Point on a Rolling Wheel

A wheel of radius r rolls along a flat surface. A point P on the rim traces the *cycloid*:

$$x(\theta) = r(\theta - \sin\theta), y(\theta) = r(1 - \cos\theta)$$

$$Find \frac{dy}{dx} \text{ and the slope at } \theta = \pi/2.$$

$$\boldsymbol{\frac{dx}{d\theta} = r(1 - \cos\theta); \frac{dy}{d\theta} = r \cdot \sin\theta:}$$

$$\boldsymbol{\frac{dy}{dx} = \sin\frac{\theta}{1 - \cos\theta}:}$$

$$\boldsymbol{At\ \theta = \frac{\pi}{2}: \frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{2}\right)}{1 - \cos\left(\frac{\pi}{2}\right)} = 1/(1 - 0) = 1:}$$

Answer: $\frac{dy}{dx} = \sin\theta/(1 - \cos\theta)$ — *the slope at* $\theta = \pi/2$ *is* 1 (45° from horizontal).

Note: At θ = 0 and θ = 2π (the contact points with the ground), dy/dx = 0/0 (indeterminate), reflecting vertical tangents — the cycloid has cusps at the contact points.

Example 37: Ellipse — Parametric Form

An ellipse is given by $x = 5\cos t, y = 3\sin t. Find \frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$.

$$\boldsymbol{\frac{dx}{dt} = -5\sin t; \frac{dy}{dt} = 3\cos t:}$$

$$\boldsymbol{\frac{dy}{dx} = 3\cos\frac{t}{-5\sin t} = -\left(\frac{3}{5}\right)\cot t:}$$

$$\boldsymbol{\frac{d}{dt}\left[\frac{dy}{dx}\right] = (3/5)\csc^2 t:}$$

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{\left(\frac{3}{5}\right)\csc^2 t}{-5\sin t} = -(3/25)\csc^3 t:}$$

At $t = \frac{\pi}{4}: \frac{dy}{dx} = -\frac{3}{5} \cdot 1 = -\frac{3}{5}$. The slope at the point (5/√2, 3/√2) on the ellipse is −3/5.

Example 38: Engineering — Connecting Rod Path

In a slider-crank mechanism, the crankpin traces:

$$x(\theta) = R \cdot \cos\theta, y(\theta) = R \cdot \sin\theta$$

(pure circle for the crankpin; R = crank radius). The velocity of the crankpin at constant angular velocity ω ($so\ \theta = \omega t$) is:

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = -R \cdot \sin\theta \cdot \omega:$$

$$\frac{dy}{dt} = R \cdot \cos\theta \cdot \omega:$$

$$Speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = R\omega\sqrt{(\sin^2\theta + \cos^2\theta)} = R\omega\ (constant\ \checkmark):$$

The crankpin moves at constant speed $R\omega$ around the circle — the direction changes but the magnitude of velocity is fixed. This fundamental result underpins balancing of rotating machinery.

Exercises 3.9 — Parametric Differentiation

Find dy/dx for each parametric curve:

1. $x = t^2 + 1, y = t^3 - t\ [find\ slope\ at\ t = 2]$
1. $x = e^t \cdot \cos t, y = e^t \cdot \sin t$
2. $x = \ln t, y = t^2 - 2t + 1\ [find\ slope\ at\ t = e]$
3. $x = 2\cos^3 t, y = 2\sin^3 t\ (astroid)\ [find\ slope\ at\ t = \pi/4]$

Applied:

1. A robotic arm tip has position $x(t) = L_1 \cdot \cos(\omega_1 t) + L_2 \cdot \cos(\omega_2 t), y(t) = L_1 \cdot \sin(\omega_1 t) + L_2 \cdot \sin(\omega_2 t)\ with\ L_1 = 0.3, L_2 = 0.2\ m, \omega_1 = 2, \omega_2 = 3$ rad/s. $Find\ dx/dt, dy/dt, and\ the\ speed\ at\ t = 0.$
1. A projectile follows $x = v_0 \cdot \cos(\alpha) \cdot t, y = v_0 \cdot \sin(\alpha) \cdot t - (g/2)t^2$. Find dy/dx and the angle of the trajectory at $t = 1\ s\ for\ v_0 = 20\ m/s, \alpha = 30°$.

3.10 Related Rates

Related rates problems arise when two or more quantities are changing simultaneously and are related by a geometric or physical equation. By differentiating this equation with respect to time t, we obtain an equation relating the rates of change — allowing us to find an unknown rate from a known one.

3.10.1 Problem-Solving Strategy

1. Identify all quantities that are changing with time and assign variable names.

1. Write down the equation relating these quantities (from geometry, physics, or engineering).
2. Differentiate both sides of the equation with respect to t, using the Chain Rule for composite expressions.
3. Substitute the given values of quantities and known rates.
4. Solve for the unknown rate.

Example 39: Rising Water Level in a Conical Tank

Water drains into an inverted conical tank (half-angle 30°, so radius r = h·tan30° = h/√3 at height h). If water enters at 2 m³/min, how fast is the water level rising when h = 3 m?

$$\boldsymbol{Volume\ of\ cone}: \boldsymbol{V} = (\frac{\mathbf{1}}{\mathbf{3}})\boldsymbol{\pi r^2 h} = (\frac{\mathbf{1}}{\mathbf{3}})\boldsymbol{\pi}(\boldsymbol{h}/\sqrt{\mathbf{3}})^{\mathbf{2}}\boldsymbol{h} = \boldsymbol{\pi h^3/9}:$$

$$\boldsymbol{Differentiate\ w.r.t.\ t}: \frac{\boldsymbol{dV}}{\boldsymbol{dt}} = (\boldsymbol{\pi}/\mathbf{3})\boldsymbol{h}^{\mathbf{2}} \cdot (\frac{\boldsymbol{dh}}{\boldsymbol{dt}}):$$

$$\boldsymbol{Given}\ \frac{\boldsymbol{dV}}{\boldsymbol{dt}} = \mathbf{2}\ \boldsymbol{m^3/min}, \boldsymbol{h} = \mathbf{3}\ \boldsymbol{m}: 2 = (\pi/3)(9) \cdot (\frac{dh}{dt})$$

$$\frac{\boldsymbol{dh}}{\boldsymbol{dt}} = \mathbf{2} \cdot \mathbf{3}/(\mathbf{9}\boldsymbol{\pi}) = \mathbf{6}/(\mathbf{9}\boldsymbol{\pi}) = \mathbf{2}/(\mathbf{3}\boldsymbol{\pi}) \approx \mathbf{0.212}\ \boldsymbol{m/min}:$$

Answer: The water level is rising at $2/(3\pi) \approx 0.212\ m/min$ *when* $h = 3\ m$.

Example 40: Expanding Oil Spill — Circular Area

An oil spill forms a thin circular film spreading at $dA/dt = 50\ m^2/min$. How fast is the radius increasing when r = 20 m?

$$\boldsymbol{A = \pi r^2 \rightarrow dA/dt = 2\pi r \cdot (dr/dt)}:$$

$$\mathbf{50} = \mathbf{2}\boldsymbol{\pi}(\mathbf{20}) \cdot \left(\frac{\boldsymbol{dr}}{\boldsymbol{dt}}\right) \rightarrow \frac{\boldsymbol{dr}}{\boldsymbol{dt}} = \mathbf{50}/(\mathbf{40}\boldsymbol{\pi}) = \mathbf{5}/(\mathbf{4}\boldsymbol{\pi}) \approx \mathbf{0.398}\ \boldsymbol{m/min}:$$

Engineering interpretation: The rate at which the spill's boundary expands decreases as the spill grows — for a given dA/dt, the boundary advances more slowly as r increases (factor 1/(2πr)). This is important for response planning: containment booms must be deployed early when the boundary is still advancing quickly.

Example 41: Radar Tracking — Aircraft Approach

An aircraft approaches a radar station at 300 m altitude, flying horizontally at 200 m/s toward the station. When the aircraft is 500 m horizontally from the station, how fast is the radar's elevation angle θ changing?

$$\textbf{Let } x = \textbf{horizontal distance}, h = 300\,m\,(\textbf{constant}). \textbf{Then } \tan\theta = h/x:$$

$$\textbf{Differentiate}: \sec^2\theta \cdot \left(\frac{d\theta}{dt}\right) = -h/x^2 \cdot \left(\frac{dx}{dt}\right):$$

$$\textbf{At } x = 500: \tan\theta = 300/500 = 0.6 \rightarrow \sec^2\theta = 1 + \tan^2\theta = 1.36:$$

$$\frac{dx}{dt} = -200\,m/s\,(\textbf{approaching}):$$

$$1.36 \cdot \left(\frac{d\theta}{dt}\right) = -300/250000 \cdot (-200) = 0.24:$$

$$\frac{d\theta}{dt} = 0.24/1.36 \approx 0.176\,rad/s:$$

Answer: The elevation angle is increasing at 0.176 rad/s ≈ 10.1°/s.

Exercises 3.10 — Related Rates

Set up and solve each related rates problem:

1. A spherical balloon is being inflated at 10 cm³/s. How fast is the radius increasing when r = 5 cm? ($V = 4\pi r^3/3$)

1. A ladder 5 m long leans against a vertical wall. The base slides away at 0.5 m/s. How fast does the top slide down when the base is 3 m from the wall?

2. Sand falls from a conveyor onto a conical pile whose height always equals its base diameter. If sand falls at 0.1 m³/min, how fast is the height growing when h = 2 m?

3. Two resistors R_1 and R_2 are in parallel: $1/R = 1/R_1 + 1/R_2$. If R_1 is increasing at 2 Ω/s and R_2 is fixed at 10 Ω, how fast is R changing when R_1 = 5 Ω?

3.11 Linear Approximation, Differentials, and Error Propagation

If f is differentiable at x = a, the tangent line at (a, f(a)) is the best linear approximation to f near a. For x near a:

Linear Approximation

$f(x) \approx f(a) + f'(a) \cdot (x - a)$ (Linear Approximation / Tangent Line Approximation)

In differential notation, writing $\Delta x = x - a \; and \; \Delta y = f(x) - f(a)$:

Differentials

$$\Delta y \approx f'(a) \cdot \Delta x$$

The differential is: $dy = f'(x) \cdot dx$

Physical meaning: dy is the change in y corresponding to an exact change dx in x,

under the approximation that the function is linear.

3.11.1 Error Propagation

In engineering measurements, a quantity y = f(x) is computed from a measured quantity x with measurement uncertainty $\pm\Delta x$. The resulting uncertainty in y is approximately:

Error Propagation via Differentials

Absolute error: $|\Delta y| \approx |f'(x)| \cdot |\Delta x|$

Relative error: $|\frac{\Delta y}{y}| \approx \left|\frac{f'(x)}{f(x)}\right| \cdot |\Delta x|$

Percentage error: $\approx \left|\frac{f'(x)}{f(x)}\right| \cdot |\Delta x| \times 100\%$

Example 42: Linear Approximation of √x

Approximate √(4.08) using the linear approximation to $f(x) = \sqrt{x} \; at \; a = 4$.

$$\boldsymbol{f(a) = \sqrt{4} = 2;\; f'(x) = 1/(2\sqrt{x}) \rightarrow f'(4) = 1/4}:$$

$$\boldsymbol{f(4.08) \approx f(4) + f'(4) \cdot (0.08) = 2 + (0.25)(0.08) = 2 + 0.02 = 2.02}:$$

Exact value: √4.08 ≈ 2.01990... (error ≈ 0.000098 — very small).

Example 43: Error Propagation in Power Calculation

Electrical power is $P = V^2/R$. A voltmeter reads $V = 120\,V \pm 0.5\,V \; and \; R = 50\,\Omega$ (exact). Find the absolute and percentage uncertainty in P.

$$\boldsymbol{P = \frac{V^2}{R} \rightarrow \frac{dP}{dV} = \frac{2V}{R}}:$$

$$\boldsymbol{\Delta P \approx |2V/R| \cdot |\Delta V| = (2 \times 120/50) \times 0.5 = 4.8 \cdot 0.5 \times 2 = 4.8 \times 0.5}: = 2.4\,W$$

$$\boldsymbol{P = 120^2/50 = 288\,W}:$$

Relative error = ΔP/P = 2.4/288 ≈ 0.00833 = 0.83%:

Alternative: Since $P \propto V^2$, *a* 1% error in V propagates to a 2% error in P (relative errors add for products). Here $\frac{\Delta V}{V} = 0.5/120 \approx 0.417\%, giving\ \Delta P/P \approx 2 \times 0.417\% = 0.833\%$ ✓

Example 44: Engineering — Temperature Measurement Error in Thermocouple

A type-K thermocouple output is V = 0.041·T mV (linear approximation). If the voltmeter has uncertainty ±0.01 mV, find the uncertainty in temperature T.

$$\boldsymbol{T = \frac{V}{0}.041 \rightarrow \frac{dT}{dV} = \frac{1}{0}.041 \approx 24.4\,°C/mV}:$$

$$\boldsymbol{\Delta T \approx 24.4 \times 0.01 = 0.244\,°C}:$$

Answer: The temperature measurement uncertainty is approximately ±0.24°C — acceptable for most process control applications (typical setpoint tolerances are ±1–2°C).

Exercises 3.11 — Linear Approximation and Differentials

Compute each linear approximation:

1. Approximate $\sqrt[3]{(8.12)}\ using\ f(x) = \sqrt[3]{x}\ at\ a = 8.$
1. Approximate $sin(31°)\ using\ f(x) = sin\,x\ at\ a = \pi/6.$
2. Approximate $e^{0.03}\ using\ f(x) = e^{x}\ at\ a = 0.$

Error Propagation:

1. The period of a simple pendulum is $T = 2\pi\sqrt{(L/g)}$. If L = 1.00 m ± 0.005 m, find ΔT.
1. The volume of a sphere is $V = (4/3)\pi r^3. If\ r = 5\ cm \pm 0.1\ mm$, find the absolute and percentage error in V.
2. Resistance $R = \rho L/A$ where ρ is resistivity, $L = 2.0\,m \pm 1\,mm, A = 1.0\,mm^2 \pm 0.02\,mm^2$. Find the percentage error in R.

3.12 Comprehensive Engineering Applications

This section presents extended engineering problems that require selecting and combining multiple differentiation techniques. Each solution clearly identifies which rules are applied and explains the physical meaning of the result.

3.12.1 Electrical Engineering

Example 45: RL Circuit — Zero-Input Response

When a fully charged inductor (initial current Io) discharges through a resistor R, the current follows $i(t) = I_0 \cdot e^{-\frac{Rt}{L}}$. The power dissipated in the resistor is $P(t) = i^2(t) \cdot R$.

Find (a) di/dt, (b) dP/dt, (c) the time at which power dissipation is half its initial value.

(a) $\boldsymbol{\frac{di}{dt} = I_0 \cdot (-\frac{R}{L} = -(\frac{R}{L}) \cdot i(t)}$: (self-decay — Chain Rule)

(b) $\boldsymbol{P(t) = RI_0^2 e^{-\frac{2Rt}{L}} \rightarrow \frac{dP}{dt} = RI_0^2 \cdot (-\frac{2R}{L}) \cdot e^{-\frac{2Rt}{L}} = -(\frac{2R}{L})P(t)}$:

(c) $\boldsymbol{P = P_0/2\ when\ e^{-\frac{2Rt}{L}} = 1/2 \rightarrow t = L \cdot ln2/(2R)}$:

Physical insight: The power decays with time constant $L/(2R)$ — half the time constant of the current decay. This makes physical sense: since $P \propto i^2$, it halves twice as fast as i.

3.12.2 Mechanical Engineering

Example 46: Rotating Shaft — Torsional Stress Gradient

A solid circular shaft of radius r transmits torque T. The shear stress at radius r from the axis is $\tau(\rho) = T \cdot \rho/(J)\ where\ J = \pi r^4/2$ is the polar moment of inertia. For a shaft where the torque varies along its length as $T(x) = T_0 \cdot cos(\pi x/(2L)), find\ d\tau/dx$ at the midpoint x = L for ρ = r.

$$\boldsymbol{\tau(x) = T(x) \cdot \frac{\rho}{J} = T_0 \cdot cos(\frac{\pi x}{2L}) \cdot \rho/(\frac{\pi r^4}{2}) = (2T_0\rho/\pi r^4) \cdot cos(\pi x/2L)}:$$

$$\boldsymbol{d\tau/dx = (2T_0\rho/\pi r^4) \cdot (-\pi/(2L)) \cdot sin(\pi x/2L) = -T_0\rho/(r^4 L) \cdot sin(\pi x/2L)}:$$

$$\boldsymbol{At\ x = L: sin(\pi/2) = 1 \rightarrow d\tau/dx = -T_0\rho/(r^4 L)}:$$

Interpretation: The magnitude of the torsional stress gradient at x = L is Toρ/(r⁴L) Pa/m. This quantity governs crack propagation risk in fatigued shafts.

3.12.3 Civil Engineering — Cable Catenary

Example 47: Shape and Tension of a Hanging Cable

A heavy cable hangs in the shape of a catenary:

$$y(x) = a \cdot cosh(x/a)\ where\ a = T_0/(\rho g)$$

(To = horizontal tension, ρ = linear density, g = 9.81 m/s²). Find dy/dx (the slope angle) and d^2y/dx^2 (curvature). Show that the arc length derivative satisfies $ds/dx = cosh(x/a)$.

$$\boldsymbol{y'(x) = a \cdot sinh(\frac{x}{a}) \cdot (\frac{1}{a}) = sinh(\frac{x}{a})}: (tangent\ angle\ \theta: tan\ \theta = sinh(\frac{x}{a}))$$

$$\boldsymbol{y''(x) = cosh(\frac{x}{a}) \cdot (\frac{1}{a}) = cosh(\frac{x}{a})/a}:$$

Arc length element: $\boldsymbol{d\frac{s}{dx} = \sqrt{(1 + (y')^2)} = \sqrt{(1 + sinh^2(\frac{x}{a}))} = \sqrt{(cosh^2(\frac{x}{a}))} = cosh\left(\frac{x}{a}\right)}$ ✓:

The tension at any point: T = To·cosh(x/a) (increases away from midspan) ✓:

Engineering note: The catenary appears in suspension bridge cables, overhead power lines, and the Gateway Arch in St. Louis (which is actually an inverted catenary). The result $ds/dx = y''$ confirms the elegant self-consistency of the catenary shape.

3.12.4 Chemical Engineering — Heat Exchanger Temperature Profile

Example 48: Temperature Gradient in a Counterflow Heat Exchanger

In a counterflow heat exchanger, the temperature of the hot fluid along the exchanger length x is approximately:

$$T_h(x) = T_h, in - (T_h, in - T_h, out) \cdot (1 - e^{-NTU \cdot \frac{x}{L}}) / (1 - e^{-NTU})$$

$$where\ NTU = UA/(\dot{m}_h \cdot cp)\ is\ the\ Number\ of\ Transfer\ Units. Simplify\ as:$$

$$T_h(x) = A + B \cdot e^{-\frac{kx}{L}}\ where\ A, B, k\ are\ constants$$

$$Find\ dT_h/dx\ and\ the\ local\ heat\ flux\ q''(x) = U \cdot (T_h(x) - T_c(x))\ assuming\ T_c(x) = C + D \cdot e^{-\frac{kx}{L}}.$$

$$\boldsymbol{dT_h/dx = B \cdot (-\frac{k}{L}) \cdot e^{-\frac{kx}{L}} = -(\frac{Bk}{L}) \cdot e^{-\frac{kx}{L}}}:$$

$$\boldsymbol{Similarly \frac{dT_c}{dx} = -(\frac{Dk}{L}) \cdot e^{-\frac{kx}{L}}}:$$

$$\boldsymbol{Local\ temperature\ difference: \Delta T(x) = (T_h - T_c) = (A - C) + (B - D)e^{-\frac{kx}{L}}}:$$

$$\boldsymbol{q''(x) = U \cdot \Delta T(x) = U[(A - C) + (B - D)e^{-\frac{kx}{L}}]:}$$

$$\boldsymbol{dq''/dx = U(B - D) \cdot (-k/L) \cdot e^{-\frac{kx}{L}}:}$$

Engineering use: The sign of dq''/dx tells engineers whether local heat transfer intensity is increasing or decreasing along the exchanger — critical for managing thermal stress and preventing hot spots.

3.13 Comprehensive Exercises and Selected Solutions

3.13.1 Mixed Differentiation — Cumulative Practice

Mixed Exercises — All Rules

State the rule(s) and differentiate:

1. $f(x) = (x^4 + 1)^6 \cdot e^{3x}$
1. $g(x) = \sin x \,/\, (1 + \cos x)$ $[QR;\ simplify\ using\ \tan(x/2)]$
2. $h(x) = \ln(x^3 + e^{2x})$
3. $k(x) = \arctan(4x)$ $[use\ d/dx[\arctan u] = u'/(1 + u^2)]$
4. $m(x) = x^{\tan x}$ $[log\ differentiation]$
5. $n(x) = (3\cos(2x) + 4\sin(x))^3$
6. $p(x) = e^{x^2 \cdot \sin x}$ $[Chain + Product\ in\ exponent]$
7. $q(x) = \sqrt{[(2x + 1)/(3x - 1)]}$ $[Quotient\ inside\ Chain]$
8. $r(x) = x^2 \cdot \arcsin(\sqrt{x})$ $[Product + Chain]$
9. $s(x) = (2x + 3)/(x^2 + x + 1);\ find\ where\ s'(x) = 0$
10. $t(x) = \ln(\sin^2 x + \cos^4 x)$ $[Chain + Sum\ inside\ log]$
11. $u(x) = \cos(x \cdot e^{x})$ $[Chain;\ inner\ function\ is\ product]$

Implicit Differentiation:

1. $x^3 \sin y + y^3 \cos x = \pi;\ find \frac{dy}{dx}$
1. $e^{x+2y} = x^3 - y^2;\ find \frac{dy}{dx}$
2. $\ln(x + y) = x^2 - y;\ find \frac{dy}{dx}\ and \frac{d^2y}{dx^2}$

Parametric:

1. $x = t - \sin t, y = 1 - \cos t\ (cycloid);\ find\ dy/dx\ at\ t = \pi/3$

1. $x = 2\cosh t, y = 3\sinh t\ (hyperbola);\ find \frac{dy}{dx}\ and\ show\ it\ satisfies\ 9x^2 - 4y^2 = 36$

3.13.2 Engineering Challenge Problems

Challenge Problems

C1 — Strain Gauge (Mech/Civil Eng): A strain gauge bonded to a steel beam reads strain $\varepsilon(M) = \frac{M}{EI}$ where M = bending moment, $E = 200\ GPa, I = 8 \times 10^{-6}\ m^4. If\ M(x) = P \cdot x \cdot (L - x)/L\ with\ P = 10\ kN, L = 5\ m, find\ d\varepsilon/dx\ at\ x = 2\ m.$

C2 — Satellite Orbit Speed (Aero/Mech Eng): Orbital speed $v = \sqrt{(GM/r)}\ where\ GM = 3.986 \times 10^{14}\ m^3/s^2, r = orbital\ radius. If\ r\ increases\ at\ 5\ m/s\ (due\ to\ atmospheric\ drag\ acting\ in\ reverse), find\ dv/dt\ at\ r = 6.8 \times 10^6\ m.$

C3 — Non-Newtonian Fluid Shear (Chem Eng): A power-law fluid has shear $stress\ \tau = K \cdot (du/dy)^n\ where\ K = 0.5\ Pa \cdot s^n, n = 0.8, and\ the\ velocity\ profile\ is\ u(y) = U \cdot \left(\frac{y}{H}\right)^{\frac{1}{n}}. Find\ d\tau/dy\ and\ evaluate\ at\ y = H/2, H = 0.02\ m, U = 0.5\ m/s.$

C4 — Antenna Gain Pattern (Elec Eng): The far-field electric field of a half-wave dipole is $E(\theta) = E_0 \cdot \cos(\pi/2 \cdot \cos\theta)/\sin\theta. Find\ dE/d\theta$ and determine the angles where $dE/d\theta = 0$ (confirm the main lobe maximum is at θ = π/2).

C5 — Reactor Concentration (Chem Eng): In a tubular reactor, concentration C(z) satisfies $C'(z) + k \cdot C(z) = 0, giving\ C(z) = C^0 e^{-kz}$. The reaction rate is $r(z) = k \cdot C(z)$. Find dr/dz and the total reaction rate $dR/dt = \int_0^L r(z)\ dz$ differentiated with respect to k.

3.13.3 Selected Full Solutions

Selected Solutions

Exercises 3.1

A1. $f'(x) = 96x^7 - 30x^5 + 16x^3 - 4x$

A3. $h(x) = x^3 - 3x + x^{-2} \rightarrow h'(x) = 3x^2 - 3 - 2x^{-3} = 3x^2 - 3 - 2/x^3$

B1. v(t) = −9.8t+20 = 0 → t = 20/9.8 ≈ 2.04 s (max height)

C1. $[(x+h)^4 - x^4]/h \;=\; [x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4]/h \;=\; 4x^3 + 6x^2h + 4xh^2 + h^3 \;\to\; 4x^3 \; as \; h \to 0$ ✓

Exercises 3.2

A2. $g'(x) = \left(\frac{2}{3}\right)x^{-\frac{1}{3}} \cdot ln\, x \;+\; x^{\frac{2}{3}} \cdot \left(\frac{1}{x}\right) = \frac{2ln\, x}{3x^{\frac{1}{3}}} + \; x^{-\frac{1}{3}} \;=\; x^{-\frac{1}{3}}(2ln\, x/3 \;+\; 1)$

A6. $\frac{d}{dx}[sin\, x \cdot sin\, x] \;=\; cos\, x \cdot sin\, x \;+\; sin\, x \cdot cos\, x \;=\; 2sin\, x \cdot cos\, x \;=\; sin\, 2x$ ✓

Exercises 3.3

A1. $f'(x) \;=\; [(4x^3 - 3)(2x^2 + 5) - (x^4 - 3x + 1)(4x)] \,/\, (2x^2 + 5)^2 \;=\; (4x^4 + 20x^3 - 6x^2 - 15 - 4x^4 + 12x^2 - 4x)/(2x^2 + 5)^2 \;=\; (6x^2 + 20x^3 - 4x - 15)/(2x^2 + 5)^2$

B2 (Michaelis-Menten). $\frac{dr}{dS} \;=\; [V_{\max}(K_{\mathrm{m}} + S) \;-\; V_{\max} \cdot S]/(K_{\mathrm{m}} + S)^2 \;=\; V_{\max} \cdot K_{\mathrm{m}}/(K_{\mathrm{m}} + S)^2$ — always positive since all terms positive ✓

Exercises 3.4

A1. $f'(x) \;=\; 12(6x^5 - 6x^2)(x^6 - 2x^3 + 5)^{11} \;=\; 72x^2(x^3 - 1)(x^6 - 2x^3 + 5)^{11}$

A4. $k'(x) \;=\; -sin(cos\, x) \cdot (-sin\, x) \;=\; sin\, x \cdot sin(cos\, x)$

C3 (arcsin). $y = arcsin\, x \;\to\; x = sin\, y \;\to\; 1 = cos\, y \cdot y' \;\to\; y' = 1/cos\, y \;=\; 1/\sqrt{(1 - sin^2 y)} \;=\; 1/\sqrt{(1 - x^2)}$ ✓

Exercises 3.6

B1. $s(t) = t^4/12 - 2t^3 + 9t^2 - 12t;\; v = t^3/3 - 6t^2 + 18t - 12;\; a = t^2 - 12t + 18;\; a = 0 \; at \; t = (12 \pm \sqrt{(144 - 72)})/2 = (12 \pm \sqrt{72})/2 = 6 \pm 3\sqrt{2};\; j = 2t - 12$

Exercises 3.9

A1. $dx/dt = 2t, dy/dt = 3t^2 - 1;\; dy/dx = (3t^2 - 1)/(2t);\; at \; t = 2: dy/dx = 11/4 = 2.75$

Exercises 3.10

A1. $V = (4/3)\pi r^3 \;\to\; dV/dt = 4\pi r^2(dr/dt) \;\to\; dr/dt = 10/(4\pi \times 25) = 1/(10\pi) \approx 0.0318 \; cm/s$

A4. $1/R = 1/R_1 \;+ 1/R_2 \;\to\; -dR/dt/R^2 = -(dR_1/dt)/{R_1}^2 \;\to\; dR/dt \;=\; R^2(dR_1/dt)/{R_1}^2$

$$R \;=\; R_1R_2/(R_1 + R_2) = 5 \times 10/15 = 10/3 \; \Omega;\; dR/dt = (10/3)^2 \times 2/25$$
$$= (100/9) \times (2/25) = 8/9 \approx 0.889 \; \Omega/s$$

Challenge C2

C2. $v = \sqrt{(GM/r)} = (GM)^{\frac{1}{2}} \cdot r^{-\frac{1}{2}}$

$$dv/dt \;=\; (GM)^{\frac{1}{2}} \cdot \left(-\frac{1}{2}\right) r^{-\frac{3}{2}} \cdot (dr/dt) \;=\; -(1/2) \cdot v/r \cdot (dr/dt)$$

$$= \; -(1/2) \times \sqrt{(3.986 \times 10^{14}/6.8 \times 10^{6})}/6.8 \times 10^{6} \times 5$$

$$= \; -(1/2) \times 7659/(6.8 \times 10^{6}) \times 5 \;\approx\; -2.82 \times 10^{-3}\; m/s^{2}$$

Speed decreases at 2.82 mm/s per second as the orbit rises.

Chapter 3 Summary

This chapter has built the complete fundamental toolkit of differential calculus. The following key achievements have been made:

1. **Sections 3.1–3.4:** Every differentiation rule (Constant, Power, Sum, Difference, Scalar Multiple, Product, Quotient, Chain) was derived rigorously from the limit definition of the derivative. Each derivation shows exactly why the rule holds and what conditions it requires.

2. **Section 3.5:** The central-difference quotient was introduced as an $O(h^2)$ numerical verification tool for any analytical derivative, with convergence tables confirming its accuracy on all major rules.

3. **Section 3.6:** Higher-order derivatives up to $f^{(n)}$ were computed and connected to physical quantities: velocity, acceleration, jerk, beam curvature, bending moment, and shear force. General nth-derivative formulas were tabulated for the standard functions.

4. **Section 3.7:** Implicit differentiation was developed as a systematic application of the Chain Rule to equations F(x,y)=0, including the computation of second derivatives. Applications included conic sections, thermodynamic equations of state, and circuit equations.

5. **Section 3.8:** Logarithmic differentiation handled variable-exponent functions (x^x, $(sin\ x)^{cos\ x}$) and complex products, and was used to derive $d/dx[a^x] = a^x ln\ a$ and $d/dx[log_a\ x] = 1/(x\ ln\ a)$ for all valid bases.

6. **Section 3.9:** Parametric differentiation $dy/dx = (dy/dt)/(dx/dt)$ was derived from the Chain Rule and applied to cycloids, ellipses, and mechanical mechanisms.

7. **Section 3.10:** Related rates problems were formulated by differentiating physical/geometric relationships with respect to time, connecting quantities that change simultaneously.

8. **Section 3.11:** Linear approximation $f(x) \approx f(a) + f'(a)(x-a)$ and differential notation $dy = f'(x)dx$ were developed, with application to error propagation in engineering measurements.

9. **Sections 3.12–3.13:** Comprehensive engineering applications across electrical, mechanical, civil, and chemical engineering demonstrated the universal utility of differentiation in professional practice.

Looking Ahead — Chapter 4 and Chapter 5

Chapter 4 extends the differentiation toolkit to transcendental functions: trigonometric, exponential, logarithmic, inverse trigonometric, and hyperbolic functions, with full first-principles derivations of each.

Chapter 5 covers **higher-order derivatives**: nth derivatives, the Leibniz product rule, concavity analysis, beam deflection theory, and vibration analysis.

Chapter 6 develops multivariable differentiation: partial derivatives, the gradient, directional derivatives, the multivariable chain rule, and the vector operators of field theory.

3.A Extended Worked Examples — Deep Dive Problems

This supplementary section provides extended, fully solved problems that combine multiple differentiation rules in a single problem. These problems reflect the level of complexity encountered in engineering examinations and professional practice.

3.A.1 Complete Analysis of a Damped Exponential Function

We perform a complete differential analysis of $f(x) = x^2 \cdot e^{-x}$ — finding all derivatives, stationary points, inflection points, and asymptotic behaviour. This function appears in the study of critically-damped systems and probability density functions.

Example 49: Complete Analysis of $f(x) = x^2 \cdot e^{-x}$

Step 1 — First Derivative (Product Rule + Chain Rule):

$$f(x) = x^2 \cdot e^{-x}:$$

$$f'(x) = 2x \cdot e^{-x} + x^2 \cdot (-1) \cdot e^{-x}:$$

$$= e^{-x} \cdot (2x - x^2) = x \cdot e^{-x} \cdot (2 - x)$$

Step 2 — Stationary Points:

$$f'(x) = 0 \text{ when } x = 0 \text{ or } x = 2 \ (\text{since } e^{-x} > 0 \text{ always}):$$

$$f(0) = 0;\ f(2) = 4 \cdot e^{-2} \approx 0.541:$$

Step 3 — Second Derivative:

$$f''(x) = d/dx\left[e^{(-x)(2x-x^2)}\right]:$$

$$= -e^{-x}(2x - x^2) + e^{-x}(2 - 2x):$$

$$= e^{-x}(-2x + x^2 + 2 - 2x) = e^{-x}(x^2 - 4x + 2)$$

Step 4 — Nature of Stationary Points:

$$f''(0) = 1 \cdot (0 - 0 + 2) = 2 > 0 \rightarrow x = 0 \textit{ is a LOCAL MINIMUM}:$$

$$f''(2) = e^{-2}(4 - 8 + 2) = -2e^{-2} < 0 \rightarrow x = 2 \textit{ is a LOCAL MAXIMUM}:$$

Step 5 — inflection Points:

$$f''(x) = 0: x^2 - 4x + 2 = 0 \rightarrow x = (4 \pm \sqrt{8})/2 = 2 \pm \sqrt{2}:$$

$$x_1 = 2 - \sqrt{2} \approx 0.586, x_2 = 2 + \sqrt{2} \approx 3.414:$$

Step 6 — Asymptotic Behaviour:

$$As\ x \rightarrow +\infty: e^{-x} \rightarrow 0\ faster\ than\ x^2\ grows \rightarrow f(x) \rightarrow 0:$$

$$As\ x \rightarrow -\infty: e^{-x} \rightarrow +\infty, x^2 \rightarrow +\infty \rightarrow f(x) \rightarrow +\infty:$$

Engineering context: The function $f(x) = x^2e^{-x}$ (with x representing time t scaled by a time constant) models the energy envelope of a critically-damped harmonic oscillator. The maximum at x=2 gives the time of peak energy storage.

Example 50: Differentiating a Nested Rational-Exponential Expression

Differentiate $f(x) = e^{x^2}/(x^2 + 1)^2$ and find all x where f'(x) = 0.

Solution — Quotient Rule:

$$Hi = e^{x^2} \rightarrow Hi' = 2x \cdot e^{x^2}\ (Chain\ Rule):$$

$$Lo = (x^2 + 1)^2 \rightarrow Lo' = 2(x^2 + 1) \cdot 2x = 4x(x^2 + 1)\ (Chain\ Rule):$$

$$f'(x) = [2x \cdot e^{x^2} \cdot (x^2 + 1)^2 - e^{x^2} \cdot 4x(x^2 + 1)] / (x^2 + 1)^4:$$

$$= e^{x^2} \cdot 2x \cdot (x^2 + 1) \cdot [(x^2 + 1) - 2] / (x^2 + 1)^4$$

$$= e^{x^2} \cdot 2x \cdot (x^2 - 1) / (x^2 + 1)^3$$

Stationary points: $f'(x) = 0\ when\ x = 0\ or\ x^2 = 1 \rightarrow x = \pm 1.$

Note that $e^{x^2} > 0\ and\ (x^2 + 1)^3 > 0$ always, so the sign of f' is determined by $2x(x^2 - 1)$.

Interval	Sign of 2x	Sign of (x^2-1)	Sign of f'	Behaviour

$x < -1$	−	+	−	Decreasing
$-1 < x < 0$	−	−	+	Increasing
$0 < x < 1$	+	−	−	Decreasing
$x > 1$	+	+	+	Increasing

Local maxima at $x = -1$ and… wait, from sign chart: f increases then decreases at x=0: x=0 is a local maximum. At $x = \pm 1$ the function has local minima.

3.A.2 Chain Rule in Signal Processing and Control Theory

The Chain Rule appears constantly in signal processing, where signals are transformed through cascaded systems. Each system block applies a function to its input, and the overall sensitivity — how the output changes with the original input — is the product of sensitivities through each block.

Example 51: Cascaded Transfer Function Sensitivity

A signal processing chain has three stages. An input signal x passes through:

1. Stage 1: Amplifier with gain $u(x) = 5 \cdot ln(1 + x)$
2. Stage 2: Non-linear element $v(u) = u^2/(u + 1)$
3. Stage 3: Output filter $w(v) = e^{-0.1v}$

Find the overall sensitivity $\frac{dw}{dx}$ $at\ x = 1$.

Chain Rule (three layers): $\frac{dw}{dx} = (\frac{dw}{dv}) \cdot (\frac{dv}{du}) \cdot (\frac{du}{dx})$

$$\boldsymbol{\frac{du}{dx} = \frac{5}{1+x} \rightarrow\ at\ x = 1: \frac{du}{dx} = 5/2 = 2.5:}$$

$$\boldsymbol{At\ x = 1: u = 5 \cdot ln2 \approx 3.466:}$$

$$\boldsymbol{\frac{dv}{du} = [2u(u+1) - u^2 \cdot 1]/(u+1)^2 = u(u+2)/(u+1)^2:}$$

$$\boldsymbol{At\ u = 3.466: \frac{dv}{du} = 3.466 \times 5.466/(4.466)^2 = 18.944/19.945 \approx 0.9498:}$$

$$\boldsymbol{At\ u = 3.466: v = (3.466)^2/4.466 = 12.013/4.466 \approx 2.690:}$$

$$\boldsymbol{\frac{dw}{dv} = -0.1 \cdot e^{-0.1v} \rightarrow\ at\ v = 2.690: \frac{dw}{dv} = -0.1 \cdot e^{-0.269} \approx -0.0764:}$$

$$\boldsymbol{\frac{dw}{dx} = (-0.0764) \cdot (0.9498) \cdot (2.5) \approx -0.1815:}$$

Interpretation: A unit increase in the input signal at x=1 decreases the output by 0.1815 units. The Chain Rule automatically accounts for the compounding of sensitivities through all three stages — a computation that would be impossible without it.

Example 52: PID Controller — Derivative Term

A PID (Proportional-Integral-Derivative) controller computes a control signal:

$$u(t) = Kp \cdot e(t) + Ki \cdot \int e(t)dt + Kd \cdot de/dt$$

where $e(t) = r(t) - y(t)$ is the error, $r(t)$ is the setpoint, and y(t) is the plant output. For a setpoint $r(t) = 5sin(2t)$ *and plant output* $y(t) = 5sin(2t - \varphi)$ (sinusoidal response with phase lag φ = π/6), find de/dt.

$$\boldsymbol{e(t) = r(t) - y(t) = 5sin(2t) - 5sin(2t - \pi/6)}:$$

$$\boldsymbol{de/dt = 10cos(2t) - 10cos(2t - \pi/6)}:$$

$$\boldsymbol{At\ t = 0}: \boldsymbol{\frac{de}{dt} = 10cos(0) - 10cos(-\pi/6) = 10 - 10(\sqrt{3}/2) = 10(1 - \sqrt{3}/2)} \approx \boldsymbol{1.34}:$$

Engineering meaning: The derivative term de/dt provides anticipatory control. At t=0, de/dt ≈ 1.34 indicates the error is growing — the controller increases the control effort proactively. With Kd = 2, the derivative contribution is 2×1.34 = 2.68 units added to the control signal.

3.A.3 Implicit Differentiation — Advanced Engineering Curves

Example 53: Lemniscate of Bernoulli — Cam Profile

The lemniscate of Bernoulli $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ $(a > 0)$ appears as a cam profile in certain mechanical followers. Find dy/dx.

Differentiate both sides w.r.t. x:

$$\boldsymbol{LHS}: \boldsymbol{2(x^2 + y^2) \cdot (2x + 2y \cdot y') = 4(x^2 + y^2)(x + y \cdot y')}:$$

$$\boldsymbol{RHS}: \boldsymbol{2a^2(2x - 2y \cdot y') = 4a^2(x - y \cdot y')}:$$

$$\boldsymbol{Expand}: \boldsymbol{4(x^2 + y^2)x + 4(x^2 + y^2)y \cdot y' = 4a^2x - 4a^2y \cdot y'}:$$

$$\boldsymbol{Group\ y'\ terms}: \boldsymbol{y'[4(x^2 + y^2)y + 4a^2y] = 4a^2x - 4(x^2 + y^2)x}:$$

$$\boldsymbol{y' \cdot 4y[(x^2 + y^2) + a^2] = 4x[a^2 - (x^2 + y^2)]}:$$

$$\boldsymbol{y' = x[a^2 - (x^2 + y^2)] / (y[a^2 + (x^2 + y^2)])}:$$

Answer: $\frac{dy}{dx} = x(a^2 - x^2 - y^2) \,/\, (y(a^2 + x^2 + y^2))$

Check at (a, 0): Numerator: $a(a^2 - a^2 - 0) = 0$. Denominator: 0·(...) = 0/0 — indeterminate, indicating the curve has a vertical tangent (cusp) at (a, 0). ✓

Example 54: Implicit Differentiation — Thermodynamic Relations

For an adiabatic process, pressure P and volume V satisfy $PV^{\gamma} = K$ (K, γ constants). Show that $dP/dV = -\gamma P/V, then\ find\ d^2P/dV^2$.

Differentiate $PV^{\gamma} = K\ w.r.t.V$:

$$\left(\frac{dP}{dV}\right) \cdot V^{\gamma} + P \cdot \gamma V^{\gamma-1} = 0:$$

$$\frac{dP}{dV} = -\gamma P/V^{\gamma} \cdot V^{-(\gamma-1)} \ldots simplified:$$

$$\frac{dP}{dV} = -P \cdot \frac{\gamma V^{\gamma-1}}{V^{\gamma}} = -\gamma P/V \checkmark:$$

$$For\ d^2P/dV^2, differentiate\ dP/dV = -\gamma P/V\ using\ Quotient\ Rule + implicit:$$

$$\frac{d^2P}{dV^2} = -\frac{\gamma\left[\left(\frac{dP}{dV}\right) \cdot V - P \cdot 1\right]}{V^2} = -\frac{\gamma\left[V \cdot \left(-\frac{\gamma P}{V}\right) - P\right]}{V^2}:$$

$$= -\frac{\gamma[-\gamma P - P]}{V^2} = \frac{\gamma(\gamma+1)P}{V^2}$$

Answer: $d^2P/dV^2 = \gamma(\gamma+1)P/V^2$ (always positive, confirming convexity of adiabatic P-V curve)

3.A.4 Logarithmic Differentiation — Engineering Power Laws

Example 55: Colebrook Equation Sensitivity

The Darcy-Weisbach friction factor f for turbulent πpe flow satisfies the Colebrook equation (implicit in f):

$$1/\sqrt{f} = -2 \cdot log_{10}(\varepsilon/(3.7D) + 2.51/(Re \cdot \sqrt{f}))$$

A simplified explicit approximation (Swamee-Jain) is:

$f(Re) = 0.25/[log_{10}(\varepsilon/(3.7D) + 5.74/Re^{0.9})]^2$

With ε/D = 0.001 (relative roughness), find df/dRe.

$$\textbf{Let } A = \frac{\varepsilon}{3.7D} = 0.0002703, g(Re) = A + 5.74/Re^{0.9}:$$

$$f = 0.25/[log_{10}(g)]^2:$$

$$\textbf{Using log differentiation — let } y = f, \ln y = \ln(0.25) - 2\ln(log_{10} g):$$

$$(\frac{1}{y})y' = -2 \cdot (1/log_{10} g) \cdot (1/(g \cdot \ln 10)) \cdot g':$$

$$g'(Re) = 5.74 \cdot (-0.9)Re^{-1.9} = -5.166Re^{-1.9}:$$

$$\textbf{At } Re = 10^5: g = 0.0002703 + 5.74/10^{4.5} = 0.0002703 + 0.001814 = 0.002084:$$

$$log_{10}(0.002084) \approx -2.681; \ f = 0.25/7.188 \approx 0.03478:$$

$y' = f \cdot [-2/(log_{10} g \cdot \ln 10) \cdot g'/g] = 0.03478 \cdot [-2/(-2.681 \times 2.303)] \cdot (-5.166 \times 10^{-9.5})/(0.002084)$**:**

Result: $df/dRe < 0$ (friction factor decreases with increasing Reynolds number in turbulent flow ✓). This sensitivity is critical for πpe network optimization.

Example 56: Population Growth Model — Logistic Equation

The logistic population model is $P(t) = K/(1 + Ae^{-rt})$ where K is the carrying capacity, r is the growth rate, and $A = (K - P_0)/P_0$.

Find dP/dt (growth rate) and d^2P/dt^2 (acceleration of growth). Show that maximum growth rate occurs at P = K/2.

$$\frac{dP}{dt} \textbf{ by Chain Rule}:$$

$$\textbf{Let } u = 1 + Ae^{-rt} \rightarrow P = \frac{K}{u} \rightarrow \frac{dP}{du} = -\frac{K}{u^2}:$$

$$\frac{du}{dt} = -Ar \cdot e^{-rt}:$$

$$\frac{dP}{dt} = -\frac{K}{u^2} \cdot (-Ar \cdot e^{-rt}) = KAr \cdot \frac{e^{-rt}}{(1 + Ae^{-rt})^2}:$$

$$This\ can\ be\ rewritten\ using\ P\ itself.\ Since\ Ae^{-rt} = \frac{K}{P} - 1:$$

$$\frac{dP}{dt} = r \cdot P \cdot \left(1 - \frac{P}{K}\right) = \frac{rP(K - P)}{K}$$

$$This\ is\ the\ standard\ logistic\ ODE!$$

$$\frac{d^2P}{dt^2} = \frac{d}{dt}[rP(1-\frac{P}{K})] = r(\frac{dP}{dt})(1-\frac{P}{K}) + rP(-\frac{1}{K})(\frac{dP}{dt}):$$

$$= r \cdot (\frac{dP}{dt}) \cdot [1 - \frac{2P}{K}] = r^2P(1-\frac{P}{K})(1-\frac{2P}{K})$$

$$\frac{d^2P}{dt^2} = 0 \text{ when } P = \frac{K}{2} \rightarrow \text{maximum growth rate at } P = \frac{K}{2} \checkmark:$$

Engineering application: This model applies to the adoption of new technologies, spread of electrical grid connections, saturation of communication channels, and bacterial growth in bioreactors.

3.A.5 Related Rates — Extended Engineering Problems

Example 57: Heat Conduction — Stefan-Boltzmann Radiation

A steel billet cools by radiation. The radiated power per unit area follows the Stefan-Boltzmann law: $P = \sigma\varepsilon \cdot T^4\, W/m^2, where\ \sigma = 5.67 \times 10^{-8}\, W/m^2K^4\ and\ \varepsilon = 0.8$ (emissivity). The billet has surface area $A = 0.5\, m^2$. Find the rate of temperature change dT/dt given that the net power lost is $Q_{loss} = A \cdot P$.

$$Q_{loss} = A \cdot \sigma\varepsilon \cdot T^4 \rightarrow dQ_{loss}/dt = A \cdot \sigma\varepsilon \cdot 4T^3 \cdot (\frac{dT}{dt}):$$

$$\text{The billet's thermal energy loss: } Q_{loss} = -m \cdot cp \cdot (\frac{dT}{dt}):$$

$$With\ m = 200\, kg, cp = 500\, J/kgK:$$

$$m \cdot cp \cdot (\frac{dT}{dt}) = -A \cdot \sigma\varepsilon \cdot 4T^3 \cdot (\frac{dT}{dt}) \ldots rearranging:$$

$$\frac{dT}{dt} = -A \cdot \sigma\varepsilon \cdot T^4 / (m \cdot cp):$$

$$At\ T = 1200\, K: \frac{dT}{dt} = -0.5 \times 5.67 \times 10^{-8} \times 0.8 \times (1200)^4/(200 \times 500):$$

$$= -0.5 \times 5.67 \times 10^{-8} \times 0.8 \times 2.0736 \times 10^{12} / 100000$$

$$= -0.5 \times 5.67 \times 0.8 \times 2.0736 \times 10^4/10^5 \approx -0.5 \times 7.497 \approx -3.75\, K/s$$

Interpretation: At 1200 K (bright orange heat), the billet cools at 3.75°C per second by radiation alone — engineers must account for this rapid cooling when scheduling reheating intervals in rolling mills.

Example 58: Hydraulic Accumulator — Pressure Rate

A hydraulic accumulator contains gas (nitrogen) at pressure P above oil. The gas obeys the adiabatic law $PV^{\gamma} = C\ with\ \gamma = 1.4$. The accumulator has total volume $V_{total} = 10\ litres$, and the oil occupies volume V_{oil}. If oil flows out at $\frac{dV_{oil}}{dt} = 0.5\frac{L}{s}, find\ \frac{dP}{dt}\ when\ P = 200\ bar, V_{oil} = 4\ L.$

$$\boldsymbol{V_{gas} = V_{total} - V_{oil} = 10 - 4 = 6\ L}:$$

$$\boldsymbol{PV_{gas}^{\gamma} = C \rightarrow differentiate\ w.r.t.t}:$$

$$\boldsymbol{\left(\frac{dP}{dt}\right)V_{gas}^{\gamma} + P \cdot \gamma V_{gas}^{\gamma-1} \cdot \left(\frac{dV_{gas}}{dt}\right) = 0}:$$

$$\boldsymbol{\frac{dV_{gas}}{dt} = -\frac{dV_{oil}}{dt} = +0.5\ L/s\ (gas\ expands\ as\ oil\ flows\ out)}:$$

$$\boldsymbol{\frac{dP}{dt} = -P \cdot \gamma / V_{gas} \cdot \left(\frac{dV_{gas}}{dt}\right) = -200 \times 1.4/6 \times 0.5}:$$

$$= -200 \times 1.4 \times 0.5/6 \approx -23.3\ bar/s$$

Interpretation: The accumulator pressure drops at 23.3 bar/s when discharging at 0.5 L/s. With an initial charge of 200 bar and minimum working pressure of 160 bar, the accumulator can supply oil for (200–160)/23.3 ≈ 1.72 s — a key parameter for emergency hydraulic system sizing.

3.A.6 Error Propagation in Engineering Measurements

Example 59: Multi-Variable Error Propagation — pipe Flow

The volumetric flow rate through a pipe is $Q = \pi \cdot D^2 \cdot v/4$ where D = diameter and v = average velocity. If D = 0.1 m ± 0.001 m and v = 3 m/s ± 0.05 m/s, find the uncertainty in Q.

$$\boldsymbol{Using\ differentials}:$$

$$\boldsymbol{Q = \left(\frac{\pi}{4}\right)D^2 v}:$$

$$\boldsymbol{dQ = \left(\frac{\pi}{4}\right)(2D \cdot dD \cdot v + D^2 \cdot dv)}:$$

$$\boldsymbol{Total\ uncertainty: \Delta Q \approx \left|\frac{\partial Q}{\partial D}\right| \cdot \Delta D + \left|\frac{\partial Q}{\partial v}\right| \cdot \Delta v}:$$

$$\boldsymbol{\frac{\partial Q}{\partial D} = \left(\frac{\pi}{4}2Dv = \left(\frac{\pi}{2}0.1 \cdot 3 = 0.4712\ m^2/s\right.\right.}:$$

$$\boldsymbol{\partial Q/\partial v = (\pi/4) \cdot D^2 = (\pi/4) \cdot 0.01 = 0.007854\ m^2}:$$

$$\Delta Q \approx 0.4712 \times 0.001 + 0.007854 \times 0.05 = 0.000471 + 0.000393 = 0.000864\ m^3/s:$$

$$Q = (\frac{\pi}{4})(0.1)^2(3) = 0.02356\ m^3/s:$$

$$Relative\ uncertainty = 0.000864/0.02356 \approx 3.67\%:$$

Observation: The diameter uncertainty contributes 0.000471/0.000864 ≈ 54.5% of the total uncertainty, and velocity contributes 45.5%. Reducing diameter measurement error would give the greatest improvement in Q accuracy.

Example 60: Sensitivity Analysis — Heat Transfer Coefficient

The Dittus-Boelter correlation for turbulent convection is:

$$Nu = 0.023 \cdot Re^{0.8} \cdot Pr^{0.4}$$

where $Nu = \frac{hD}{k}\ (Nusselt), Re = \frac{\rho vD}{\mu}\ (Reynolds), Pr = \frac{\mu cp}{k}\ (Prandtl)$. Find the sensitivity of Nu to a 1% change in Re.

Taking the logarithmic derivative:

$$ln(Nu) = ln(0.023) + 0.8 \cdot ln(Re) + 0.4 \cdot ln(Pr):$$

$$\frac{d(ln\ Nu)}{d(ln\ Re)} = 0.8 \rightarrow (\frac{dNu}{Nu})/(\frac{dRe}{Re}) = 0.8:$$

$$A\ 1\%\ increase\ in\ Re\ gives\ a\ 0.8\%\ increase\ in\ Nu\ (and\ hence\ in\ h):$$

$$This\ means: to\ double\ the\ heat\ transfer\ coefficient\ h, we\ need\ to\ increase\ Re\ by$$

$$(2^{\frac{1}{0.8}} - 1) \times 100\% \approx 123\%:$$

Design implication: Doubling the convective heat transfer coefficient requires more than doubling the flow velocity (since Re ∝ v). Engineers seeking to enhance heat transfer should focus on increasing turbulence intensity (e.g. through surface roughness) rather than simply increasing flow rate.

3.A.7 Complete Reference Table of Standard Derivatives

The following table compiles all standard derivatives needed for engineering calculations. Mastery of this table, combined with the combination rules (Product, Quotient, Chain), enables differentiation of virtually any elementary function encountered in practice.

Complete Standard Derivatives Reference

f(x)	f′(x)	Domain / Conditions
c	0	$c\ constant$
x^n	nx^{n-1}	$n \in \mathbb{R}, x \neq 0\ if\ n < 0$
e^x	e^x	$x \in \mathbb{R}$
a^x	$a^x \cdot \ln a$	$a > 0, a \neq 1$
$\ln x$	$1/x$	$x > 0$
$\log_a x$	$1/(x \cdot \ln a)$	$x > 0, a > 0, a \neq 1$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\sec^2 x$	$x \neq \pi/2 + n\pi$
$\cot x$	$-\csc^2 x$	$x \neq n\pi$
$\sec x$	$\sec x \cdot \tan x$	$x \neq \pi/2 + n\pi$
$\csc x$	$-\csc x \cdot \cot x$	$x \neq n\pi$
$\arcsin x$	$1/\sqrt{(1-x^2)}$	$\lvert x\rvert < 1$
$\arccos x$	$-1/\sqrt{(1-x^2)}$	$\lvert x\rvert < 1$
$\arctan x$	$1/(1+x^2)$	$x \in \mathbb{R}$
$\operatorname{arccot} x$	$-1/(1+x^2)$	$x \in \mathbb{R}$
$\sinh x$	$\cosh x$	$x \in \mathbb{R}$
$\cosh x$	$\sinh x$	$x \in \mathbb{R}$
$\tanh x$	$\operatorname{sech}^2 x$	$x \in \mathbb{R}$
$\operatorname{arcsinh} x$	$1/\sqrt{(x^2+1)}$	$x \in \mathbb{R}$
$\operatorname{arccosh} x$	$1/\sqrt{(x^2-1)}$	$x > 1$
$\operatorname{arctanh} x$	$1/(1-x^2)$	$\lvert x\rvert < 1$
$\lvert x\rvert$	$x/\lvert x\rvert\ =\ sgn(x)$	$x \neq 0$

3.B Final Examination-Style Practice Problems

The following problems are at examination level and require confident selection and combination of all rules covered in this chapter. Work each problem completely, checking your answer numerically where possible.

Final Examination Practice — Complete Set

Section A — Pure Differentiation (2 marks each):

y = $(3x^2 - 1)^8 \cdot e^{4x};\ find\ dy/dx$

y = $[sin(x^2 + 1)]^{\frac{1}{3}};\ find\ dy/dx$

y = $ln(arctan\ x);\ find\ dy/dx$

y = $x^{e^x};\ find\ dy/dx$

y = $\sqrt{[(1 + x)/(1 - x)]};\ find\ dy/dx$

y = $cosh(x^2) \cdot sin(e^x);\ find\ dy/dx$

Find $f''(x)\ for\ f(x)\ =\ x \cdot sin\ x$

Differentiate implicitly: $x \cdot e^y\ +\ y \cdot e^x\ =\ 1$

Section B — Applied Problems (4 marks each):

The position of a particle in the x-y plane is given parametrically by $x(t) = 2t - sin(2t), y(t) = 1 - cos(2t)$. Find the speed $|v|\ =\ \sqrt{((dx/dt)^2 + (dy/dt)^2)}$ as a function of t. At what times is the speed zero?

The volume flow rate through a rectangular weir is $Q\ =\ (2/3) \cdot Cd \cdot \sqrt{(2g)} \cdot L \cdot H^{\frac{3}{2}}$, where $Cd = 0.62, L = 2\ m, g = 9.81\ m/s^2$. If the head H is rising at $\frac{dH}{dt} = 0.02\frac{m}{s}, find\frac{dQ}{dt}\ when\ H = 0.5\ m.$

A rectangular beam of width w and depth d is cut from a circular log of radius r ($so\ w^2 + d^2 = 4r^2$). The bending strength is proportional to S=w·d². Using implicit differentiation, find the value of d/w that maximises S for a given log radius.

The impedance of a series RLC circuit is $Z(\omega) = \sqrt{(R^2 + (\omega L - 1/(\omega C))^2)}$. Find $dZ/d\omega$ and show it equals zero at the resonance frequency $\omega_0 = 1/\sqrt{(LC)}$.

A chemical reactor produces product P at rate $dP/dt\ =\ k(T) \cdot (A_0 - P)^2$ where $k(T) = 0.1 \cdot e^{-\frac{5000}{T}}, T = 350\ K, A_0 = 2\ mol/L, P = 0.5$ mol/L initially. Find dP/dt and d²P/dt² = $d/dt[k \cdot (A_0 - P)^2]$ using the Chain and Product Rules.

Section C — Proof and Derivation (5 marks each):

From the limit definition, prove the Product Rule: $d/dx[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. State clearly what assumption you make about the continuity of g.

Using logarithmic differentiation, prove $d/dx[x^n]\ =\ n \cdot x^{n-1}$ for all real n. Hence derive $\frac{d}{dx}\left[\sqrt{x}\right] and\ \frac{d}{dx}[\frac{1}{x}]$.

For a parametric curve $x = r \cdot cos\ \theta, y = r \cdot sin\ \theta$ (circle of radius r), show that $d^2y/dx^2 = -\frac{1}{r}$ at every point, confirming constant curvature $\kappa = 1/r$.

3.C Theoretical Foundations — Differentiability, Continuity, and Limits

This section establishes the theoretical underpinning of the differentiation rules and addresses common misconceptions about differentiability.

3.C.1 Differentiability Implies Continuity

Theorem: Differentiability → Continuity

If f is differentiable $at\ x\ =\ a$, then f is continuous at x = a.

Proof: Since f'(a) exists, we can write $f(x) - f(a) = \frac{[f(x)-f(a)]}{x-a} \cdot (x - a). As\ x \to a$, the first factor → f'(a) and the second factor → 0. Therefore $lim_{\{x\to a\}}[f(x) - f(a)] = f'(a) \cdot 0 = 0$, which means $lim_{\{x\to a\}} f(x) = f(a)$. ■

Converse is FALSE: Continuity does NOT imply differentiability. Example: $f(x) = |x|$ is continuous at x=0 but not differentiable there (the left and right derivatives are −1 and +1 respectively).

3.C.2 The Limit Definition Rewritten

The limit definition can be written in an equivalent form that is sometimes more convenient:

Alternative Form of the Limit Definition

$$f'(a) = lim_{\{x\to a\}} \frac{[f(x) - f(a)]}{x - a}$$

(This replaces the variable h = x−a; same limit, different form)

This form is useful for computing derivatives at specific points without first finding f'(x) generally.

Example 61: Using the Alternative Limit Form

Find f'(4) for f(x) = √x using the alternative limit definition.

$$f'(4) = lim_{\{x\to 4\}} [\sqrt{x} - \sqrt{4}]/(x-4):$$

Rationalise: multiply by (√x+2)/(√x+2):

$$= lim_{\{x\to 4\}} \frac{x-4}{[(x-4)(\sqrt{x}+2)]} = lim_{\{x\to 4\}} \frac{1}{\sqrt{x}+2} = 1/4:$$

Answer: $f'(4) = \frac{1}{(2\sqrt{4})m} = 1/4$ ✓(matches the Power Rule result)

3.C.3 Non-Differentiable Points and Engineering Significance

Functions are not differentiable at points where they are discontinuous, have corners (kinks), or have vertical tangents. Engineering functions commonly exhibit all three types:

Type of Non-Differentiability	Engineering Example
Discontinuity	Switch-mode power supply: voltage jumps at switching instants
Corner (kink)	piecewise linear approximation to stress-strain curve at yield point
Vertical tangent	Free surface of a liquid: contact angle at wall
Oscillatory discontinuity	Turbulent velocity at a point: not differentiable in time

When a physical model produces a non-differentiable function, engineers must use one-sided derivatives, generalised derivatives (in the sense of distributions), or numerical finite differences as appropriate. Recognising when a function is not differentiable is as important as being able to differentiate it when it is.

3.C.4 Leibniz Rule for Differentiation Under the Integral Sign

A powerful extension of the Chain Rule known as the Leibniz Rule allows differentiation of integrals whose limits depend on the differentiation variable:

Leibniz Rule (Differentiation Under the Integral Sign)

$$\frac{d}{dx}[\int_{a(x)}^{b(x} f(t,x)\,dt] = f(b(x),x)\cdot b'(x) - f(a(x),x)\cdot a'(x) + \int_{a(x)}^{b(x} \frac{\partial f}{\partial x}\,dt$$

This rule, named after Leibniz but often attributed to Feynman who popularised it as a 'trick', appears in heat transfer (differentiation of temperature integrals), fluid mechanics (Reynolds Transport Theorem), and control theory (sensitivity of integrals to parameters).

Example 62: Simple Leibniz Rule Application

Find $\frac{d}{dx}[\int_0^x t^2 \cdot sin(xt)\,dt]$.

Using Leibniz rule with $a = 0, b = x, f(t,x) = t^2 sin(xt)$:

$$\frac{d}{dx} = f(x,x)\cdot 1 - f(0,x)\cdot 0 + \int_0^x \partial/\partial x\ [t^2 sin(xt)]dt:$$

$$= x^2 sin(x^2) + \int_0^x t^3 cos(xt)\ dt:$$

Answer: $= x^2 \cdot sin(x^2) + \int_0^x t^3 \cdot cos(xt)dt$

The remaining integral cannot be simplified further without knowing x, but the differentiation itself is correctly performed by Leibniz's rule.

3.D Answers to Selected Final Examination Problems

Selected Answers — Section A

A1. $\frac{dy}{dx} = (3x^2-1)^7 \cdot e^{4x} \cdot [48x^2 - 8 + 4(3x^2-1)] = e^{4x}(3x^2-1)^7(24x^2 + 8x - \ldots =$ $8(3x^2-1)^7 \cdot e^{4x} \cdot (3x + \ldots$

Full: $dy/dx = 8e^{4x} \cdot (3x^2-1)^7 \cdot (3x^2 + 4x - 1)\ldots$ verify by expanding Product Rule fully:

Exact: $dy/dx = e^{4x} \cdot (3x^2-1)^7 \cdot [8 \cdot 6x + (3x^2-1) \cdot 4 \cdot 8] \ldots = 8e^{4x}(3x^2-1)^7[6x \ldots$ product rule properly:

Properly: Let $u = (3x^2-1)^8, v = e^{4x};\ u' = 16x(3x^2-1)^7, v' = 4e^{4x}$

$$\frac{dy}{dx} = 16x(3x^2-1)^7 \cdot e^{4x} + (3x^2-1)^8 \cdot 4e^{4x} = 4e^{4x}(3x^2-1)^7[4x + (3x^2-1)]$$
$$= 4e^{4x}(3x^2-1)^7(3x^2 + 4x - 1)$$

A3. $y = ln(arctan\, x) \rightarrow y' = (1/arctan\, x) \cdot 1/(1 + x^2)$

A4. $y = x^{e^x}$: $ln\, y = e^x \cdot ln\, x \rightarrow \frac{y'}{y} = e^x \cdot ln\, x + \frac{e^x}{x} \rightarrow y' = x^{e^x} \cdot e^x(ln\, x + \frac{1}{x})$

A7. $f(x) = x \cdot \sin x \rightarrow f' = \sin x + x \cos x \rightarrow f''$

$$= cos\, x + cos\, x - x\, sin\, x = 2cos\, x - x\, sin\, x$$

A8. $xe^y \cdot (y') + e^y + ye^x + e^x \cdot y' = 0 \rightarrow y'(xe^y + e^x)$

$$= -e^y - ye^x \rightarrow y' = -\frac{e^y + ye^x}{xe^y + e^x}$$

Selected Answers — Section B

B1. $\frac{dx}{dt} = 2 - 2cos(2t), \frac{dy}{dt} = 2sin(2t);\ |v|^2 = (2 - 2cos2t)^2 + 4sin^2 2t$

$$= 4(1-cos2t)^2 + 4sin^2 2t = 4[1 - 2cos2t + cos^2 2t + sin^2 2t] = 4[2 - 2cos2t]$$
$$= 8(1 - cos2t)$$

$$|v| = \sqrt{(8(1-cos2t))} = 2\sqrt{2} \cdot \sqrt{(1-cos2t)} = 4|sin\, t| \; (using\; 1 - cos2t = 2sin^2 t)$$

$$Speed = 0\; when\; sin\, t = 0, i.e., t$$
$$= n\pi\; (n\; integer) —\; these\; are\; the\; cusp\; points\; of\; the\; cycloid. \checkmark$$

B4 (RLC resonance). $Z = \sqrt{(R^2 + X^2)}\; where\; X = \omega L - \frac{1}{\omega C}$

$$\frac{dZ}{d\omega} = X \cdot \frac{\frac{dX}{d\omega}}{Z} where \frac{dX}{d\omega} = L + 1/(\omega^2 C)$$

$$Set\; dZ/d\omega = 0: X = 0 \rightarrow \omega L = 1/(\omega C) \rightarrow \omega^2 = 1/(LC) \rightarrow \omega_0 = 1/\sqrt{(LC)} \checkmark$$

3.E Detailed Worked Solutions to Core Exercises

This section provides fully worked solutions to a representative selection of exercises from each section, demonstrating the thought process and algebraic care required for accurate differentiation.

3.E.1 Solutions to Exercises 3.1 — Basic Rules

Example 63: Full Solution: Exercises 3.1 A3 — Dividing and Differentiating

Differentiate $h(x) = (x^5 - 3x^3 + 2x) / x^2$.

Step-by-step approach:

Step 1 — Divide each term by x²:

$$h(x) = x^5/x^2 - 3x^3/x^2 + 2x/x^2 = x^3 - 3x + 2x^{-1}$$

Step 2 — Apply Power Rule to each term:

$$h'(x) = 3x^2 - 3 + 2 \cdot (-1)x^{-2} = 3x^2 - 3 - 2/x^2$$

Step 3 — Check reasonableness:

At $x = 1: h(1) = (1 - 3 + 2)/1 = 0$. The function has x=1 as a root. $h'(1) = 3 - 3 - 2 = -2$, so the function is declining at $x = 1$. This is geometrically consistent with a function that crosses zero and turns negative nearby.

Answer: $h'(x) = 3x^2 - 3 - 2/x^2$

Example 64: Full Solution: Exercises 3.1 B2 — Marginal Cost

Cost function: $C(x) = 0.004x^3 - 0.8x^2 + 100x + 5000$. $Find\ C'(x)\ at\ x = 80$.

$$\boldsymbol{C'(x) = 0.004 \cdot 3x^2 - 0.8 \cdot 2x + 100 = 0.012x^2 - 1.6x + 100}:$$

$$\boldsymbol{C'(80) = 0.012 \cdot 6400 - 1.6 \cdot 80 + 100 = 76.8 - 128 + 100}$$
$$\boldsymbol{= 48.8\ KES\ '000\ per\ tonne}:$$

Economic interpretation: At a production level of 80 tonnes, the marginal cost is KES 48,800 per additional tonne. Since the average cost is $C(80)/80 = (0.004 \cdot 512000 - 0.8 \cdot 6400 + 8000 + 5000)/80 = (2048 - 5120 + 8000 + 5000)/80 = 9928/80 \approx KES\ 124{,}100$ per tonne, the marginal cost is well below the average cost, suggesting the operation is in the efficient scale range and could benefit from expansion.

3.E.2 Solutions to Exercises 3.2 — Product Rule

Example 65: Full Solution: Product Rule — Exercises 3.2 B1

Spring-mass: $x(t) = (1 + 0.2t) \cdot cos(5t)$. $Find \frac{dx}{dt}\ at\ t = \pi/10\ s$.

$$\boldsymbol{Let\ u = 1 + 0.2t \rightarrow u' = 0.2};$$

$$\boldsymbol{Let\ v = cos(5t) \rightarrow v' = -5sin(5t)}:$$

$$\boldsymbol{\frac{dx}{dt} = 0.2cos(5t) + (1 + 0.2t) \cdot (-5sin(5t))}:$$

$$\boldsymbol{At\ t = \pi/10}: \boldsymbol{5t = \pi/2, cos(\pi/2) = 0, sin(\pi/2) = 1, 1 + 0.2(\pi/10) = 1 + \pi/50}$$
$$\boldsymbol{\approx 1.0628}:$$

$$\boldsymbol{\frac{dx}{dt} = 0.2 \cdot 0 + (-5)(1.0628) \cdot 1 = -5.314\ m/s}:$$

Physical meaning: At t = π/10 s (quarter period of the 5 rad/s oscillation), the spring is at maximum extension and the mass is moving back toward equilibrium at 5.314 m/s. The amplitude is slightly larger than 1 because the envelope (1+0.2t) is still growing at this time.

3.E.3 Solutions to Exercises 3.3 — Quotient Rule

Example 66: Full Solution: Michaelis-Menten Kinetics

$r(S) = V_{\max} \cdot S/(K_{\mathrm{m}} + S)$. $Prove\ dr/dS > 0$ always.

$$\boldsymbol{\frac{dr}{dS} = [V_{\max} \cdot (K_{\mathrm{m}} + S) - V_{\max} \cdot S \cdot 1] / (K_{\mathrm{m}} + S)^2}:$$

$$= V_{\max} \cdot K_{\mathrm{m}} / (K_{\mathrm{m}} + S)^2$$

Since $V_{\max} > 0, K_{\mathrm{m}} > 0, and\ S \geq 0$, all factors in the numerator and denominator are positive. Therefore $dr/dS > 0\ for\ all\ S \geq 0$. ✓

As $S \to \infty$: $dr/dS = V_{\max}K_{\mathrm{m}}/(K_{\mathrm{m}} + S)^2 \to 0$. The reaction rate saturates and becomes insensitive to further increases in substrate concentration — a key enzyme saturation effect.

As $S \to 0$: $dr/dS \to V_{\max}/K_{\mathrm{m}}$, the maximum sensitivity. Small additions of substrate near zero concentration give the maximum incremental rate increase.

3.E.4 Solutions to Exercises 3.4 — Chain Rule

Example 67: Full Solution: Exercises 3.4 C1 — Temperature Gradient

$T(x) = T_0 \cdot e^{-\beta x^2}, \beta = 0.08$. Find $dT/dx\ at\ x = 2\ m$.

Inner: $\boldsymbol{u = -\beta x^2 = -0.08x^2 \to u' = -0.16x}$:

Outer: $\boldsymbol{e^u \to e^u}$:

$$\boldsymbol{dT/dx = T_0 \cdot e^{-0.08x^2} \cdot (-0.16x)}:$$

$$\boldsymbol{At\ x = 2: dT/dx = T_0 \cdot e^{-0.32} \cdot (-0.32) = -0.32T_0 \cdot e^{-0.32}}:$$

$$\boldsymbol{If\ T_0 = 200°C: dT/dx = -0.32 \times 200 \times 0.7261 = -46.5°C/m}:$$

Physical interpretation: The temperature is dropping at 46.5°C per metre at x=2 m from the heat source. This steep gradient drives rapid heat conduction by Fourier's law $q = -k \cdot \frac{dT}{dx}$. For copper (k=400 W/mK), the heat flux is 400×46.5 = 18,600 W/m² — a very high thermal load.

3.E.5 Multi-Step Engineering Problem — Full Worked Solution

Example 68: Comprehensive: Optimal Insulation Thickness

Heat loss per unit length from a πpe of inner radius r1=0.05 m, outer radius r2, insulated to outer radius r3, is:

$$Q/L = 2\pi(T_i - T_o) \,/\, [ln(r_2/r_1)/k_\pi pe + ln(r_3/r_2)/k_{ins} + 1/(h_o \cdot r_3)]$$

where $k_\pi pe = 50\ W/mK, k_{ins} = 0.04\ W/mK, h_o = 10\ W/m^2K, T_i - T_o = 200°C, r_2 = 0.055\ m$. Treating Q/L as a function of r3 alone:

$$\boldsymbol{Let\ A = ln(r_2/r_1)/k_\pi pe + 1/k_ins \cdot ln(r_3/r_2) + 1/(h_o \cdot r_3)}:$$

$$\boldsymbol{Numerically: A = ln(1.1)/50 + 25 \cdot ln(r_3/0.055) + 0.1/r_3}:$$

$$\boldsymbol{= 0.001906 + 25 \cdot ln(r_3/0.055) + 0.1/r_3}:$$

$$\boldsymbol{Q/L = 400\pi/A}:$$

$$\boldsymbol{d(Q/L)/dr_3 = -400\pi \cdot A'/(A^2)}:$$

$$\boldsymbol{A' = 25/r_3 - 0.1/{r_3}^2}:$$

$$\boldsymbol{Set\ A' = 0: 25/r_3 = 0.1/{r_3}^2 \rightarrow r_3 = 0.1/25 = 0.004\ m}:$$

This gives r3 < r2, which is physically impossible (insulation cannot have negative thickness).

This means $A'(r_3) > 0$ for all $r_3 > r_2$ ($since\ 25/r_3 > 0.1/{r_3}^2\ when\ r_3 > 0.004$), so Q/L is *decreasing* with increasing r3 — adding more insulation always reduces heat loss (for these parameters). This confirms that there is no finite 'optimal' insulation thickness in this case; the minimum heat loss is achieved with maximum feasible insulation.

Critical radius concept: For a hollow cylinder, there exists a 'critical radius' $r_c = k_{ins}/h_o = 0.04/10 = 0.004\ m$. When $r_2 > r_c$ ($as\ here: r_2 = 0.055\ m >> 0.004\ m$), adding insulation always reduces heat loss. When $r_2 < r_c$, adding insulation initially *increases* heat loss (the 'insulation paradox' — extra insulation enlarges the outer surface area, increasing convective loss faster than it reduces conductive resistance).

3.E.6 Derivation of Inverse Trigonometric Derivatives

The inverse trigonometric functions appear frequently in engineering: arctan arises in phase calculations, arcsin in geometric optics and robotics, and arccos in structural analysis. All their derivatives are derived using implicit differentiation.

Example 69: Deriving $\boldsymbol{d/dx[arcsin\ x] = 1/\sqrt{(1-x^2)}}$

Let $y = arcsin\ x, so\ x = sin\ y, -\pi/2 \leq y \leq \pi/2$.

Differentiate x = sin y w.r.t. $\boldsymbol{x}$: $1 = cos\ y \cdot (dy/dx)$

Solve: $\boldsymbol{dy/dx = 1/cos\ y}$:

Since $\boldsymbol{cos\ y \geq 0\ on\ [-\pi/2, \pi/2]\ and\ sin^2 y + cos^2 y = 1}$:

$$\boldsymbol{cos\ y = \sqrt{(1 - sin^2 y)} = \sqrt{(1-x^2)}}:$$

$$\boldsymbol{dy/dx = 1/\sqrt{(1-x^2)}\ \checkmark}:$$

$$d/dx[arcsin\ x] = 1/\sqrt{(1-x^2)}\ (|x| < 1)$$

Example 70: Deriving $\boldsymbol{d/dx[arctan\ x] = 1/(1+x^2)}$

Let $y = arctan\ x, so\ x = tan\ y, -\pi/2 < y < \pi/2$.

Differentiate: $\mathbf{1 = sec^2 y \cdot (dy/dx)}$:

$$\mathbf{dy/dx = cos^2 y}:$$

$$\mathbf{Since\ tan\ y = x: sec^2 y = 1 + tan^2 y = 1 + x^2, so\ cos^2 y = 1/(1 + x^2)}:$$

$$\mathbf{dy/dx = 1/(1 + x^2)}\ \checkmark:$$

$$\frac{d}{dx}[arctan\ x] = 1/(1 + x^2)\ (x \in \mathbb{R})$$

All Inverse Trigonometric Derivatives

$\boldsymbol{Function\ y = f(x)}$	$\boldsymbol{dy/dx}$
$arcsin\ x$	$1/\sqrt{(1 - x^2)}, \lvert x\rvert < 1$
$arccos\ x$	$-1/\sqrt{(1 - x^2)}, \lvert x\rvert < 1$
$arctan\ x$	$1/(1 + x^2), x \in \mathbb{R}$
$arccot\ x$	$-1/(1 + x^2), x \in \mathbb{R}$
$arcsec\ x$	$1/(\lvert x\rvert\sqrt{(x^2 - 1)}), \lvert x\rvert > 1$
$arccsc\ x$	$-1/(\lvert x\rvert\sqrt{(x^2 - 1)}), \lvert x\rvert > 1$

3.E.7 Engineering Applications — Electrical Systems

Example 71: Instantaneous Frequency of a Chirp Signal

In radar and sonar, a chirp signal has a phase that increases quadratically with time:

$$s(t) = A \cdot cos(\varphi(t))\ where\ \varphi(t) = 2\pi(f_0 t + ½ \cdot k \cdot t^2)$$

The instantaneous frequency is defined as f_i (t) = (1/2π)·dφ/dt.

$$\mathbf{d\varphi/dt = 2\pi(f_0 + k \cdot t)}:$$

$$\mathbf{f_i(t) = f_0 + k \cdot t}:$$

The frequency increases linearly from fo at t=o to fo+kT at t=T. This linear frequency sweep is the defining characteristic of a linear frequency modulated (LFM) chirp — the foundation of modern radar pulse compression.

For $\mathbf{f_0 = 1\ GHz, k = 10^6\ Hz/s, T = 1\ ms}$:

$$\mathbf{Bandwidth = kT = 10^6 \times 10^{-3} = 1000\ Hz\ (1\ kHz\ swept\ in\ 1\ ms)}:$$

Engineering use: The derivative $d\varphi/dt$ gives instantaneous frequency — a concept only possible because of differential calculus. Range resolution of the $radar \approx c/(2B) =$

$3 \times 10^8/(2 \times 1000) = 150\, km\ per\ Hz$, demonstrating that wider bandwidth (faster frequency sweep) gives finer range resolution.

Example 72: Transmission Line Voltage Gradient

For a lossy transmission line, the voltage along the line (in the phasor domain) follows:

$$V(z) = V^+ \cdot e^{-\gamma z} + V^- \cdot e^{+\gamma z}$$

where $\gamma = \alpha + j\beta$ is the propagation constant (α = attenuation, β = phase constant). For a $50\,\Omega\ line\ with\ \alpha = 0.02\ Np/m, \beta = 2\pi/\lambda\ rad/m, \lambda = 0.1\ m\ (f = 3\ GHz)$, matched load $(V^- = 0), V^+ = 10\ V. Find\ dV/dz$ and the voltage 'gradient' at z = 0.

$$\boldsymbol{V(z) = 10 \cdot e^{-(0.02+j62.8)\,z} =:}$$

$$\boldsymbol{\frac{dV}{dz} = 10 \cdot (-0.02 - j62.8) \cdot e^{-\gamma z}:}$$

$$\boldsymbol{At\ z = 0: \frac{dV}{dz} = 10 \cdot (-0.02 - j62.8) = -0.2 - j628\ V/m:}$$

$$\boldsymbol{|dV/dz|_0 = 10 \cdot |\gamma| = 10 \cdot \sqrt{(0.0004 + 3944.8)} \approx 628\ V/m:}$$

Interpretation: The voltage changes at 628 V/m along the line — primarily due to the phase variation (β >> α at microwave frequencies). The small attenuation (0.02 Np/m) causes slow amplitude decay, while the large phase constant causes rapid phase rotation — 2π radians every 10 cm.

3.E.8 Derivatives of Hyperbolic Functions

Hyperbolic functions arise in the analysis of transmission lines, catenary curves, heat transfer in fins, and the solutions of many second-order ODEs. Their derivatives are analogous to — but not identical to — those of the trigonometric functions.

Hyperbolic Function Definitions

Definitions:

$$sinh\ x = (e^x - e^{-x})/2 \quad cosh\ x = (e^x + e^{-x})/2$$

$$tanh\ x = sinh\ x\ /\ cosh\ x \quad sech\ x = 1/cosh\ x$$

Key identity: $cosh^2 x - sinh^2 x = 1$

Example 73: Deriving Hyperbolic Derivatives

(a) $\boldsymbol{d/dx[\sinh x]}$:

$$\boldsymbol{d/dx[(e^{x} - e^{-x})/2] \;=\; (e^{x} + e^{-x})/2 \;=\; \cosh x}\ \checkmark$$

(b) $\boldsymbol{d/dx[\cosh x]}$:

$$\boldsymbol{d/dx[(e^{x} + e^{-x})/2] \;=\; (e^{x} - e^{-x})/2 \;=\; \sinh x}\ \checkmark$$

(c) $\boldsymbol{d/dx[\tanh x]}$:

$$\boldsymbol{QR}: \frac{[\cosh x \cdot \cosh x \;-\; \sinh x \cdot \sinh x]}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\ \checkmark$$

$$\frac{d}{dx}[\sinh x] \;=\; \cosh x$$

$$\frac{d}{dx}[\cosh x] \;=\; \sinh x \;(note\colon\; +\sinh, not\; -\sinh\; like\; trig!)$$

$$\frac{d}{dx}[\tanh x] \;=\; \operatorname{sech}^2 x$$

$$\frac{d}{dx}[\operatorname{sech} x] \;=\; -\operatorname{sech} x \cdot \tanh x$$

Example 74: Fin Temperature Distribution

The temperature in an extended surface (fin) satisfies

$T(x) \;=\; T_\infty \;+\; (T_b - T_\infty) \cdot \frac{\cosh(m(L-x))}{\cosh(mL)}$ *where* $m = \sqrt{\frac{hP}{kA_c}}$ is the fin parameter.

Find $\frac{dT}{dx}$ *and show* $\frac{d^2T}{dx^2} \;=\; m^2(T - T_\infty)$.

Let $\boldsymbol{\theta(x) \;=\; T(x) - T_\infty \;=\; (T_b - T_\infty) \cdot \frac{\cosh(m(L-x))}{\cosh(mL)}}$:

$$\boldsymbol{\frac{d\theta}{dx} = \frac{T_b - T_\infty}{\cosh(mL)} \cdot \sinh(m(L-x)) \cdot (-m)}\colon$$

$$= -m(T_b - T_\infty) \cdot \frac{\sinh(m(L-x))}{\cosh(mL)}$$

$$\boldsymbol{\frac{d^2\theta}{dx^2} \;=\; -\frac{m(T_b - T_\infty)}{\cosh(mL)} \cdot \cosh(m(L-x)) \cdot (-m)}\colon$$

$$= m^2 \cdot (T_b - T_\infty) \cdot \frac{\cosh(m(L-x))}{\cosh(mL)} \;=\; m^2 \cdot \theta \;=\; m^2(T - T_\infty)\ \checkmark$$

This confirms: The fin temperature satisfies the fin equation $d^2\theta/dx^2 - m^2\theta = 0$, whose general solution is $\theta = C_1 cosh(mx) + C_2 sinh(mx)$. This ODE governs not just fins but also transmission lines (Telegrapher's equations) and diffusion in membranes.

3.E.9 Connection to Numerical Methods — Newton-Raphson

The derivative finds immediate application in Newton's Method for finding roots of nonlinear equations — one of the most important algorithms in engineering computation.

Newton-Raphson Method

Newton-Raphson Iteration:

$$x_{\{n+1\}} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Geometric meaning: follow the tangent line from $(x_n, f(x_n))$ to where it crosses the x-axis.

Convergence: quadratic (doubles significant figures each iteration) near simple roots.

Example 75: Newton-Raphson Applied to a Non-Linear Equation

Find √7 by applying Newton's Method to $f(x) = x^2 - 7$, starting from x0 = 3.

$$\boldsymbol{f(x) = x^2 - 7 \rightarrow f'(x) = 2x}:$$

Iteration formula: $\boldsymbol{x_{\{n+1\}} = x_n - (x_{n^2} - 7)/(2x_n) = (x_n + 7/x_n)/2}$:

$\boldsymbol{n}$	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n) = x_{n^2} - 7}$	$\boldsymbol{f'(x_n) = 2x_n}$	$\boldsymbol{x_{\{n+1\}}}$
0	3.000000	2.000000	6.000000	2.666667
1	2.666667	0.111111	5.333333	2.645833
2	2.645833	0.000434	5.291667	2.645751
3	2.645751	0.000000007	5.291502	2.645751

After just 3 iterations starting from x_0=3, we have √7 ≈ 2.645751 — correct to 6 decimal places. The quadratic convergence is clearly visible: errors go from 0.354 → 0.021 → 0.000082 → ~3×10^{-8}, roughly squaring each time.

Engineering use: Newton's Method is used in power flow analysis (solving nonlinear power system equations), structural analysis (Newton-Raphson for nonlinear finite

element problems), CFD (pressure-velocity coupling), and control system design. Without the derivative f'(x), this algorithm would not exist.

3.F Mean Value Theorem and Its Engineering Significance

The Mean Value Theorem (MVT) is one of the deepest consequences of differentiability. It formalises the intuitive idea that a differentiable function must attain its average rate of change as an instantaneous rate somewhere on every interval.

Mean Value Theorem

If f is continuous on [a, b] and differentiable on (a, b), then there exists at least one point c ∈ (a, b) such that:

Mean Value Theorem

$$f'(c) = \frac{[f(b) - f(a)]}{b - a}$$

Geometrically: the tangent line at c is parallel to the secant line from a to b.

Example 76: MVT Applied to Velocity

A car travels along a straight road. Its position (km) at time $t\ (h)\ is\ s(t) = 4t^3 - 3t^2 + 2t. Between\ t = 1\ h\ and\ t = 3\ h, the\ average\ speed\ is$:

$$\boldsymbol{s(3) - s(1) = (108 - 27 + 6) - (4 - 3 + 2) = 87 - 3 = 84\ km\ in\ 2\ h \rightarrow avg = 42\ km/h}:$$

$$\boldsymbol{By\ MVT, \exists c \in (1,3): s'(c) = 42}:$$

$$\boldsymbol{s'(t) = 12t^2 - 6t + 2 = 42 \rightarrow 12t^2 - 6t - 40 = 0 \rightarrow 6t^2 - 3t - 20 = 0}:$$

$$\boldsymbol{t = (3 \pm \sqrt{(9 + 480)})/12 = (3 \pm \sqrt{489})/12}:$$

$$\boldsymbol{c \approx (3 + 22.11)/12 \approx 2.09\ h\ (the\ other\ root\ c \approx -1.59\ is\ outside\ [1,3])}:$$

Meaning: At t ≈ 2.09 hours, the car's speedometer reads exactly 42 km/h — its average speed over the full two-hour interval. The MVT guarantees this moment exists, even without knowing exactly when.

3.F.1 L'Hôpital's Rule — Preview

A direct consequence of the MVT (via Cauchy's generalised MVT) is L'Hôpital's Rule for resolving 0/0 and ∞/∞ indeterminate limits. Though its full treatment belongs in Chapter 7, we state it here since it relies entirely on differentiation:

L'Hôpital's Rule

$$L'Hôpital's\ Rule: If\ lim_{\{x\to a\}}\ f(x)/g(x)\ is\ of\ the\ form\ 0/0\ or\ \infty/\infty, then:$$

$$lim_{\{x\to a\}}\frac{f(x)}{g(x)} = \ lim_{\{x\to a\}}\frac{f'(x)}{g'(x)}$$

(provided the right-hand limit exists or is ±∞)

Example 77: L'Hôpital Preview — Classic Limit

Evaluate $lim_{\{x\to 0\}}\frac{sin(x)}{x}\ using\ L'Hôpital's\ Rule.$

Form: 0/0 at x=0:

$$\boldsymbol{Apply\ L'Hôpital: lim_{\{x\to 0\}}\frac{cos(x)}{1}\ =\ cos(0)\ =\ 1:}$$

Answer: $lim_{\{x\to 0\}}\ sin\ x\ /\ x\ =\ 1$ ✓(the fundamental trigonometric limit)

This confirms the derivative $\frac{d}{dx}[sin\ x] = \ cos\ x\ at\ x = 0, since\ \frac{[sin(x)-sin(0)]}{x-0}\ \to\ cos(0)\ =\ 1.$

Example 78: L'Hôpital — Engineering Form

In electrical engineering, the impedance of an inductor in DC conditions requires the limit:

$$lim_{\{\omega\to 0\}}\ j\cdot\omega\cdot L/(R\ +\ j\cdot\omega\cdot L)$$

Taking magnitudes: $lim_{\{\omega\to 0\}}\ \omega L\ /\ \sqrt{(R^2\ +\ \omega^2L^2)}\ \to\ 0/R\ =\ 0.$

No L'Hôpital needed here (form is 0/R, not 0/0).

But consider the quality factor $Q\ =\ \omega L/R$ at resonance $\omega_0\ =\ 1/\sqrt{(LC)}. As\ R\to 0$:

$$lim_{\{R\to 0\}}\ Q\ =\ lim_{\{R\to 0\}}\ \omega_0 L/R\ =\ +\infty$$

A lossless inductor has infinite Q — it stores energy indefinitely. The derivative $dQ/dR\ =\ -\omega_0 L/R^2\ \to\ -\infty\ as\ R\to 0$, confirming Q is extremely sensitive to small resistances in high-Q circuits.

3.F.2 Full Treatment of L'Hôpital's Rule — All Standard Forms

L'Hôpital's Rule applies not only to the 0/0 and ∞/∞ forms but, after algebraic manipulation, to five other indeterminate forms: 0·∞, ∞−∞, 0^0, 1^∞, and ∞^0. The following examples cover each form systematically.

Example 78b: 0/0 Form Requiring Repeated Application

Form 0/0. Evaluate $\lim(x\to 0)\ (e^x - 1 - x) / x^2$

At x = 0: numerator → 0, denominator → 0. Apply L'Hôpital once:

$$= \boldsymbol{lim}(\boldsymbol{x \to 0})\ (\boldsymbol{e}^x - \mathbf{1}) / (\mathbf{2}\boldsymbol{x})\ [\boldsymbol{still}\ \mathbf{0/0}] \ \to\ \boldsymbol{apply\ again}:\ = \boldsymbol{lim}(\boldsymbol{x \to 0})\ \boldsymbol{e}^x / \mathbf{2} = \mathbf{1/2}$$

Answer: 1/2. *Note:* This confirms the second-order Taylor coefficient of e^x — the result is the coefficient of x^2 in the expansion $e^x = 1 + x + x^2/2 + \cdots$, namely 1/2. L'Hôpital and Taylor series are deeply connected: both give access to the same limit information.

Example 78c: ∞/∞ Form — Exponential Dominates Polynomial

Form ∞/∞. Evaluate $\lim(x\to\infty)\ x^2 / e^x$

As x→∞: numerator → ∞, denominator → ∞. Apply L'Hôpital twice:

$$\boldsymbol{lim}\ \boldsymbol{x}^2/\boldsymbol{e}^x \ \to\ \boldsymbol{lim}\ \mathbf{2}\boldsymbol{x}/\boldsymbol{e}^x \ \to\ \boldsymbol{lim}\ \mathbf{2}/\boldsymbol{e}^x \ =\ \mathbf{0}$$

Answer: 0. Exponentials grow faster than any polynomial. In control engineering: the natural response of a stable system (pole at s = −1 giving e^{-t}) dominates any polynomial forcing after sufficient time — this limit proves it rigorously. By induction, $\lim(x\to\infty)\ x^n/e^x = 0$ for every positive integer n.

Example 78d: 0·∞ Form — Entropy of Mixing

Form 0·∞. Evaluate $\lim(x\to 0^+)\ x \cdot \ln x$

As $x\to 0^+$: x→0 and ln x→−∞, so the product is 0·(−∞). Rewrite as ∞/∞ form:

$$\boldsymbol{x}\cdot\boldsymbol{ln}\ \boldsymbol{x} \ =\ \boldsymbol{ln}\ \boldsymbol{x} / (\mathbf{1}/\boldsymbol{x}) \ \to\ \infty/\infty\ \boldsymbol{form}.\ \boldsymbol{Apply\ L'H\hat{o}pital}:\ = (\mathbf{1}/\boldsymbol{x}) / (-\mathbf{1}/\boldsymbol{x}^2) \ =\ -\boldsymbol{x} \to\ \mathbf{0}$$

Answer: 0. Engineering context: In entropy calculations for mixtures, the entropy of mixing involves terms p·ln p (where p is a mole fraction). The result $\lim(p\to 0)\ p\cdot\ln p = 0$ ensures the entropy formula remains continuous and well-defined as a component vanishes from the mixture — essential for chemical process simulation.

Example 78e: 1^∞ Form — Derivation of e from First Principles

Form 1^∞. Evaluate $\lim(n\to\infty)\ (1 + 1/n)^n$ [the definition of e]

Let $y = (1 + 1/x)^x$ (continuous version). Take ln: $\ln y = x \cdot \ln(1 + 1/x) \to 0 \cdot \infty$ form.

$$\boldsymbol{\ln y = \ln(1 + 1/x) / (1/x)} \ [\mathbf{0/0 \ as \ x \to \infty}] \ \to \ \boldsymbol{L'H\hat{o}pital}:$$
$$= [-\mathbf{1}/(\boldsymbol{x}(\boldsymbol{x} + \mathbf{1}))] / (-\mathbf{1}/\boldsymbol{x}^2) = \boldsymbol{x}^2/(\boldsymbol{x}(\boldsymbol{x} + \mathbf{1})) = \boldsymbol{x}/(\boldsymbol{x} + \mathbf{1}) \to \mathbf{1}$$

Therefore $\ln y \to 1$, so $y \to e^1 = e$. **Answer: $\lim(n\to\infty)(1+1/n)^n = e \approx 2.71828$.** This is the celebrated limit that defines Euler's number. In compound interest: investing £1 at 100% annual interest compounded n times per year grows to $(1+1/n)^n \to e$ in the limit of continuous compounding — the basis of continuous-time financial mathematics and the exponential decay/growth model in engineering.

L'Hôpital's Rule — Complete Summary of Indeterminate Forms

Indeterminate Form	Conversion Strategy	Then Apply L'Hôpital
0/0 or ∞/∞	Direct application	Yes
0·∞	Rewrite as f/(1/g) or g/(1/f) to get 0/0 or ∞/∞	Yes
∞ – ∞	Combine fractions or factor to get 0/0 or ∞/∞	Yes
0^0, 1^∞, ∞^0	Take ln: exponent becomes 0·∞ → convert → L'Hôpital	Yes (on ln y)

Warning: L'Hôpital's Rule applies only when the limit is genuinely indeterminate. Do not apply it when the denominator has a non-zero limit — this gives a wrong answer. Always verify the form (0/0 or ∞/∞) before applying.

Exercises 3.F.2 — L'Hôpital's Rule Practice

Evaluate each limit, identifying the form and applying L'Hôpital's Rule as needed:

(a) $\lim(x\to0) \tan x / x$ (b) $\lim(x\to\pi) \sin x / (x - \pi)$ (c) $\lim(x\to\infty) \ln x / \sqrt{x}$

(d) $\lim(x\to0^+) x^x$ [form 0^0; take ln first] (e) $\lim(x\to\infty) (x^2 + x) - x^2$ / (algebraically simplify first)

(f) $\lim(x\to0) (1 - \cos x) / x^2$ [important: appears in structural dynamics damping analysis]

(g) $\lim(\omega\to\omega_0) \sin(\omega\cdot t)/(\omega - \omega_0)$ [resonance limit in driven oscillator analysis; treat t as parameter]

Selected Answers: (a) 1 (b) 1 (c) 0 (d) 1 (f) 1/2 (g) $t\cdot\cos(\omega_0\cdot t)$

3.G Differentiation in Numerical Engineering — Finite Difference Methods

In many engineering problems, functions are known only at discrete data points — from measurements, simulations, or look-up tables. Differentiation must then be approximated numerically using **finite difference formulas** derived directly from the limit definition.

3.G.1 Forward, Backward, and Central Difference Formulas

Formula	***Expression***
$Forward\ difference$	$f'(x) \approx \frac{[f(x+h)-f(x)]}{h}$
$Backward\ difference$	$f'(x) \approx \frac{[f(x)-f(x-h)]}{h}$
$Central\ difference$	$f'(x) \approx \frac{[f(x+h)-f(x-h)]}{2h}$
$Second\ derivative\ (central)$	$f''(x) \approx \frac{[f(x+h)-2f(x)+f(x-h)]}{h^2}$
$Fourth-order\ central$	$f'(x) \approx \frac{[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)]}{12h}$

Example 79: Numerical Differentiation from Experimental Data

A pump test records flow rate Q (L/s) against head H (m):

$\boldsymbol{H\ (m)}$	$\boldsymbol{Q\ (L/s)}$
10	45.2
15	40.8
20	35.1
25	28.7
30	20.3

Estimate $\frac{dQ}{dH}\ at\ H\ =\ 20\ m$ using the central difference formula (h = 5 m).

$$\boldsymbol{\frac{dQ}{dH} \approx \frac{[Q(25)-Q(15)]}{2\times 5} = \frac{[28.7-40.8]}{10} = -12.1/10 = -1.21\ L/s\ per\ m}:$$

Interpretation: At H = 20 m, increasing the head by 1 m reduces the flow rate by 1.21 L/s. This $\frac{dQ}{dH}$ is the slope of the pump characteristic curve — a key parameter for system matching and stability analysis in hydraulic design.

Compare with one-sided estimates:

Forward: $\frac{[\boldsymbol{Q(25)-Q(20)}]}{\mathbf{5}} = \frac{[\mathbf{28.7-35.1}]}{\mathbf{5}} = \mathbf{-1.28\ L/s/m}$:

Backward: $\frac{[\boldsymbol{Q(20)-Q(15)}]}{\mathbf{5}} = \frac{[\mathbf{35.1-40.8}]}{\mathbf{5}} = \mathbf{-1.14\ L/s/m}$:

The central difference (−1.21) is the best estimate — it averages the forward and backward slopes, cancelling the leading-order error term.

Example 80: Numerical Second Derivative — Beam Curvature from Strain Gauge Data

Strain gauges on a beam measure deflection y (mm) at positions x (mm):

$\boldsymbol{x}$ $(\boldsymbol{mm})$	$\boldsymbol{y}$ $(\boldsymbol{mm})$
0	0.000
50	0.482
100	0.841
150	1.021
200	0.982
250	0.721
300	0.000

Estimate the curvature $\kappa \approx y''\, at\, x = 150\, mm$ using the central second-difference formula with h = 50 mm.

$$\boldsymbol{y''(150) \approx \frac{[y(200) - 2y(150) + y(100)]}{h^2}}:$$

$$= \frac{[\mathbf{0.982 - 2(1.021) + 0.841}]}{\mathbf{(50)^2}}:$$

$$= \frac{[\mathbf{0.982 - 2.042 + 0.841}]}{\mathbf{2500}}:$$

$$= -0.\frac{219}{2500} = -8.76 \times 10^{-5}\, mm^{-1}:$$

Bending moment $M = EI \cdot y'' = 200 \times 10^3\, N/mm^2 \times 5 \times 10^6\, mm^4 \times (-8.76 \times 10^{-5})\, mm^{-1}$:

$$= -87{,}600\, N \cdot mm \approx -87.6\, kN \cdot mm \approx -87.6\, N \cdot m$$

Engineering outcome: The bending moment at midspan is approximately −87.6 N·m (hogging — top fibres in compression). This numerical differentiation of measured deflection data allows engineers to infer internal forces without installing moment gauges directly.

3.H Integration of Differentiation Rules — Multi-Rule Problems

Real engineering functions rarely require just one differentiation rule. This final section provides problems whose solutions require **three or more rules applied in sequence**, reflecting the level of mathematical fluency expected in professional engineering practice.

Example 81: Five-Rule Problem — Full Decomposition

Differentiate $f(x) = ln(sin^2 x \cdot e^{x^3}) \,/\, arctan(\sqrt{x})$.

Strategy: Quotient Rule (outer), then Chain Rule + logarithm laws for numerator, Chain Rule for denominator.

Simplify numerator first: $ln(sin^2 x \cdot e^{x^3}) = 2ln(sin\, x) + x^3$:

Let $g(x) = 2ln(sin\, x) + x^3$ **and** $h(x) = arctan(\sqrt{x})$:

$$g'(x) = \frac{2cosx}{sinx} + 3x^2 = 2cot\, x + 3x^2:$$

$$h'(x) = \frac{1}{1 + \left(\sqrt{x}\right)^2} \cdot \left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x(1+x)}}:$$

$$f'(x) = \frac{[g'(x) \cdot h(x) - g(x) \cdot h'(x)]}{h(x)^2} \text{ (Quotient Rule)}:$$

$$= \frac{\left[(2cot\, x + 3x^2) \cdot arctan\left(\sqrt{x}\right) - \dfrac{2ln(sin\, x) + x^3}{2\sqrt{x(1+x)}}\right]}{\left[arctan\left(\sqrt{x}\right)\right]^2}$$

Rules used: Logarithm simplification, Quotient Rule, Chain Rule (×3: for ln(sin x), for e^{x^3} absorbed into simplification, and for arctan(√x)), and the standard derivatives of ln, sin, and arctan.

Example 82: Engineering Masterclass — Optimal Fin Geometry

The efficiency of a straight rectangular fin is:

$$\eta(m, L) = tanh(mL) / (mL)$$

where $m = \sqrt{(2h/(kt))}$ depends on fin thickness t, conductivity k, and convection coefficient h. For fixed volume $V = w \cdot L \cdot t$ ($w = width$), we have $t = V/(wL)$, so:

$$m(L) = \sqrt{(2hwL/(kV))} = C \cdot \sqrt{L}\ where\ C = \sqrt{(2hw/(kV))}$$

Hence $mL = C \cdot L^{\frac{3}{2}}. Find\ d\eta/dL$ and determine the optimal L.

Let $u = mL = C \cdot L^{\frac{3}{2}}. Then\ \eta = tanh(u)/u$:

$$\frac{d\eta}{du} = \frac{[u \cdot sech^2 u - tanh\, u]}{u^2}:$$

$$\frac{du}{dL} = (\frac{3}{2})C \cdot L^{\frac{1}{2}} = (\frac{3}{2})(u/L^{\frac{3}{2}}) \cdot L\text{^}(1/2) = 3u/(2L):$$

$$\frac{d\eta}{dL} = \left(\frac{d\eta}{du}\right) \cdot \left(\frac{du}{dL}\right) = \frac{[u \cdot sech^2 u - tanh\, u]}{u^2} \cdot \frac{3u}{2L}:$$

$$= \frac{3(u \cdot sech^2 u - tanh\, u)}{2L \cdot u}$$

Set $d\eta/dL = 0$: $u \cdot sech^2 u = tanh\, u \rightarrow u = tanh\, u/sech^2 u = sinh\, u \cdot cosh\, u$:

This transcendental equation: $u = (1/2)sinh(2u) \rightarrow$ satisfied at u=0 only (trivially).

The equation $u \cdot sech^2 u = tanh\, u$ has no solution for u > 0, meaning $d\eta/dL < 0\ for\ all\ L > 0$. The efficiency *always decreases* as L increases for fixed volume: a shorter, thicker fin is always more efficient than a longer, thinner fin of the same volume. This counter-intuitive result — derived purely through differentiation — guides practical fin design toward compact, thick geometries.

Final Exercises 3.H — Multi-Rule Integration

Each problem requires three or more rules. Identify all rules before computing:

f(x) = $arcsin(e^x) \cdot \sqrt{(1 - e^{2x})};\ identify\ the\ product\ and\ chain\ rules\ needed$

g(x) = $ln\left[(x^2 + 1)^{\frac{1}{2}} \cdot \frac{sin(2x)}{e^{3x}}\right]\ ;\ simplify\ with\ log\ laws\ first$

h(x) = $x^{arctan\,x};\ logarithmic\ differentiation$

k(x) = $sin(x \cdot ln\,x);\ chain\ +\ product\ inside\ chain$

Parametric: $x = e^t \cdot cos\ t, y = e^t \cdot sin\ t. Show \frac{dy}{dx} = \frac{sin\ t + cos\ t}{cos\ t - sin\ t}$.

Implicit: $x \cdot sin(y) + y \cdot cos(x) = 1. Find \frac{dy}{dx}\ at\ (\pi/2, \pi/2)\ if\ it\ satisfies\ this.$

Challenge:

For $f(x) = x^{x^x}$, use logarithmic differentiation twice to find f'(x). Evaluate f'(1).

Prove: if $y = (1 + x)^{\frac{1}{x}}, then\ as\ x \to 0, y \to e.$

$Then\ find \frac{dy}{dx}\ using\ logarithmic\ differentiation\ and\ show\ lim_{\{x \to 0\}} \frac{dy}{dx} = -\frac{e}{2}.$

3.I Chapter Glossary — Key Terms and Definitions

Glossary of Key Terms

Term	***Definition***
Derivative	*The limit of the difference quotient*: $f'(x) = \frac{lim_{\{h \to 0\}}[f(x+h) - f(x)]}{h}$
Differentiable	*A function f is differentiable at x if* $f'(x)$ *exists (finite limit)*
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}\ for\ all\ real\ n$
Product Rule	$\frac{d}{dx}[fg] = f'g + fg';\ rate\ of\ area\ change\ of\ rectangle$
Quotient Rule	$\frac{d}{dx}[f/g] = \frac{f'g - fg'}{g^2};\ derived\ from\ Product\ Rule$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x);\ rate\ multiplication\ for\ compositions$
Implicit differentiation	$Differentiating\ F(x,y) = 0\ w.r.t.\ x\ using\ Chain\ Rule\ on\ y - terms$
Logarithmic differentiation	$Taking\ ln\ before\ differentiating;\ handles\ variable\ exponents$

$Parametric\ differentiation$	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\ for\ curves\ x(t), y(t)$
$Related\ rates$	$Using\ \frac{d}{dt}[F(x,y,z)] = 0\ to\ connect\ rates\ of\ change$
$Linear\ approximation$	$f(x) \approx f(a) + f'(a)(x-a);\ tangent\ line\ approximation$
$Differential$	$dy = f'(x)dx;\ approximate\ change\ in\ y\ for\ change\ dx\ in\ x$
$Higher-order\ derivative$	$f^{(n)} = \frac{d}{dx}[f^{(n-1)}];\ nth\ repeated\ differentiation$
$Central\ difference$	$\frac{[f(x+h)-f(x-h)]}{2h};\ O(h^2)\ numerical\ approximation\ to\ f'(x)$
$Mean\ Value\ Theorem$	$\exists c \in (a,b): f'(c) = \frac{[f(b)-f(a)]}{b-a}\ if\ f\ differentiable\ on\ (a,b)$
$L'Hôpital's\ Rule$	$lim\frac{f}{g} = lim\frac{f'}{g'}\ when\ limit\ is\ 0/0\ or\ \infty/\infty$
$Instantaneous\ rate$	$The\ derivative\ as\ the\ limit\ of\ average\ rates;\ the\ 'speedometer'\ value$
$Curvature$	$\kappa = \lvert y'' \rvert/(1+y'^2)^{\frac{3}{2}};\ how\ rapidly\ a\ curve\ bends$
$Jerk$	$Third\ derivative\ \frac{d^3s}{dt^3};\ rate\ of\ change\ of\ acceleration$

Chapter 4 — Advanced Differentiation Techniques

Advanced Differentiation Techniques

Trigonometric • Exponential • Logarithmic • Inverse Trig • Hyperbolic Functions

---

Chapter 3 established the foundational rules of differentiation: the Power Rule, Product Rule, Quotient Rule, and Chain Rule applied to algebraic functions. This chapter extends the complete differentiation toolkit to every class of function encountered in engineering practice. Eight systematic sections cover the six trigonometric functions, natural and general exponential functions, natural and general logarithmic functions, the six inverse trigonometric functions, hyperbolic functions and their inverses, and the three powerful techniques of implicit, logarithmic, and parametric differentiation.

Every section in this chapter follows a consistent four-part structure. First, the engineering context is established — showing where and why the technique matters in practice. Second, every formula is derived rigorously from first principles: understanding the origin of a result is the surest way to apply it correctly under unfamiliar conditions. Third, a progression of fully worked examples moves from straightforward applications to engineering-level problems drawn from real contexts: vibration analysis, AC circuit theory, heat transfer, structural mechanics, robotics, chemical kinetics, and fluid dynamics. Fourth, each section closes with a graded exercise set, coded by difficulty from routine application to research-level challenge.

Fifteen mathematical diagrams are integrated throughout the chapter to build geometric and physical intuition for every technique. A complete master reference table at the end collects all rules from Chapters 3 and 4 into a single professional-grade reference.

Chapter Objectives — On completion, students will be able to:

1. Derive and apply the derivatives of all six trigonometric functions from the fundamental limit definitions.
2. Derive $\frac{d}{dx}(e^x)$ and $\frac{d}{dx}(a^x)$ from first principles and apply with the Chain Rule to all exponential forms.
3. Derive $\frac{d}{dx}(\ln x)$ and $\frac{d}{dx}(\log_a x)$ by implicit differentiation of the inverse exponential.
4. Derive and apply the derivatives of all six inverse trigonometric functions using implicit differentiation.

5. Define hyperbolic functions from exponentials, derive all their derivatives, and apply to catenary and transmission-line problems.
6. Differentiate any implicit relationship $F(x, y) = 0$ for both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
7. Apply logarithmic differentiation to products, quotients, and variable-exponent expressions (x^x, f^g).
8. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for parametrically defined curves; identify cusps, extrema, and tangent directions.
9. Select the most efficient differentiation technique for any given function and apply it to engineering problems.

4.1 Derivatives of Trigonometric Functions

Trigonometric functions are the mathematical language of periodicity. They appear in every engineering discipline: AC voltages and currents ($v = V_m \sin(\omega t + \varphi)$), mechanical vibrations ($x = A\cos(\omega t)$), structural wave propagation, rotating machinery, acoustic analysis, antenna radiation patterns, and the Fourier representation of any periodic signal. The ability to differentiate them precisely is therefore one of the most frequently used skills in engineering mathematics.

Two fundamental limits provide the entire foundation of trigonometric differentiation. They are proved geometrically by comparing the area of a circular sector of unit radius with the areas of two triangles — one inscribed and one circumscribed — and then applying the squeeze theorem:

Two Fundamental Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Both limits require x measured in radians. In degrees, the first limit would equal $\pi/180$, introducing an unwanted constant factor into every derivative. This is why radians are used exclusively in calculus.

Figure 4.1.1: $y = \sin x$ (blue) and $y = \cos x$ (gold dashed). At every maximum of $\sin x$ the slope is zero, so $\cos x = 0$ there. At every zero crossing of $\sin x$ the slope is ± 1, so $\cos x = \pm 1$. This is the visual meaning of $\frac{d}{dx}(\sin x) = \cos x$.

Figure 4.1.2: The derivative as limit of secant slopes: as $h \to 0$, the secant through $P = (a, f(a))$ and $Q = (a+h, f(a+h))$ rotates towards the tangent at P, whose slope is $f'(a)$.

4.1.1 Derivation: $\frac{d}{dx}(\sin x) = \cos x$

Begin with the definition of the derivative. Expand $\sin(x+h)$ using the sine addition formula $\sin(A+B) = \sin A\cos B + \cos A\sin B$:

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x \cdot (0) + \cos x \cdot (1) = \cos x$$

Derivative of sine

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

4.1.2 Derivation: $\frac{d}{dx}(\cos x) = -\sin x$

Use the co-function identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and the Chain Rule (inner function $= \frac{\pi}{2} - x$, derivative $= -1$):

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2}-x\right)\right] = \cos\left(\frac{\pi}{2}-x\right)\cdot(-1) = -\sin x$$

Derivative of cosine

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

4.1.3 Derivation: $\frac{d}{dx}(\tan x) = \sec^2 x$

Write $\tan x = \sin x/\cos x$ and apply the Quotient Rule:

$$\frac{d}{dx}(\tan x) = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivative of tangent

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

4.1.4 Remaining Three: cot$x$, sec$x$, csc$x$

Cotangent: write $\cot x = \cos x / \sin x$, apply Quotient Rule:

$$\frac{d}{dx}(\cot x) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

Secant: write $\sec x = (\cos x)^{-1}$, apply Chain Rule:

$$\frac{d}{dx}(\sec x) = -(\cos x)^{-2} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

Cosecant: write $\csc x = (\sin x)^{-1}$, apply Chain Rule:

$$\frac{d}{dx}(\csc x) = -(\sin x)^{-2} \cdot \cos x = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$$

Figure 4.1.3: Chain Rule schematic: x maps via g to u, then via f to y. The derivative multiplies both rates: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Figure 4.1.4: Product Rule — area interpretation. The increment $d(fg)$ has three parts: $f\,dg$ (gold), $g\,df$ (red), and $df\,dg$ (grey). As $dx \to 0$ the third term is negligible, leaving $\frac{d(fg)}{dx} = f\,g' + g\,f'$.

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

4.1.5 Chain Rule with Trigonometric Functions

When the argument is a function $u(x)$, multiply by $u'(x)$:

$$\frac{d}{dx}[\sin u] = \cos u \cdot u' \qquad \frac{d}{dx}[\cos u] = -\sin u \cdot u' \qquad \frac{d}{dx}[\tan u] = \sec^2 u \cdot u'$$

4.1.6 Worked Examples

Example 4.1.1

Differentiate: (a) $y = 5\sin(3x)$ (b) $y = x^2\cos x$ (c) $y = \tan^2 x$ (d) $y = \frac{\sin x}{1+\cos x}$

Solution:

1. Chain Rule — outer sin, inner $u = 3x$, $u' = 3$:

$$\frac{dy}{dx} = 5\cos(3x) \cdot 3 = 15\cos(3x)$$

1. Product Rule — $f = x^2$, $f' = 2x$; $g = \cos x$, $g' = -\sin x$:

$$\frac{dy}{dx} = 2x\cos x + x^2(-\sin x) = x(2\cos x - x\sin x)$$

2. Write $[\tan x]^2$. Chain Rule — outer u^2, inner $\tan x$, derivative $\sec^2 x$:

$$\frac{dy}{dx} = 2\tan x \sec^2 x$$

3. Quotient Rule — numerator $f = \sin x$, denominator $g = 1 + \cos x$:

$$\frac{dy}{dx} = \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

■

Example 4.1.2

A mechanical component oscillates: $s(t) = 0.04\sin(50\pi t)$ m. Find: (a) velocity and maximum speed; (b) acceleration at $t = 0.01$ s; (c) first time acceleration is zero.

Solution:

4. $v(t) = \frac{ds}{dt} = 0.04 \cdot 50\pi\cos(50\pi t) = 2\pi\cos(50\pi t)$ m/s.

Maximum speed $= 2\pi \approx 6.28$ m/s.

5. $a(t) = \frac{dv}{dt} = -2\pi \cdot 50\pi\sin(50\pi t) = -100\pi^2\sin(50\pi t)$

At $t = 0.01$: $a = -100\pi^2\sin(\pi/2) = -100\pi^2 \approx -986.96$ m/s².

6. $a = 0$ when $\sin(50\pi t) = 0$, i.e. $t = n/50$. First positive: $t = 1/50 = 0.02$ s.

■

Example 4.1.3

AC circuit: capacitor current $i = C\,\frac{dv}{dt}$, $v(t) = 220\sin(100\pi t)$ V, $C = 50\ \mu$F. Find $i(t)$ and peak current.

Solution:

$$\frac{dv}{dt} = 220 \cdot 100\pi\cos(100\pi t) = 22\,000\pi\cos(100\pi t)\ \text{V/s}$$

$$i(t) = C\frac{dv}{dt} = 50 \times 10^{-6} \cdot 22\,000\pi\cos(100\pi t) = 1.1\pi\cos(100\pi t)\ \text{A}$$
$$\approx 3.456\cos(314.16\,t)\ \text{A}$$

Peak current $I_m = 1.1\pi \approx 3.456$ A. Current leads voltage by 90° — the defining property of a capacitor.

■

Example 4.1.4

Prove $y = A\sin(nx) + B\cos(nx)$ satisfies $\frac{d^2y}{dx^2} + n^2y = 0$ for any constants A, B.

Solution:

$$\frac{dy}{dx} = An\cos(nx) - Bn\sin(nx)$$

$$\frac{d^2y}{dx^2} = -An^2\sin(nx) - Bn^2\cos(nx) = -n^2[A\sin(nx) + B\cos(nx)] = -n^2y$$

$$\frac{d^2y}{dx^2} + n^2y = -n^2y + n^2y = 0 \qquad \text{QED}$$

This equation governs simple harmonic oscillators, LC circuits, vibrating strings, and acoustic resonators. The general solution is a linear combination of $\sin(nx)$ and $\cos(nx)$.

■

Example 4.1.5

Find all $x \in [0,\ 2\pi]$ where the tangent to $y = 2\sin x + \sin(2x)$ is horizontal.

Solution:

$$\frac{dy}{dx} = 2\cos x + 2\cos(2x) = 2\cos x + 2(2\cos^2 x - 1) = 4\cos^2 x + 2\cos x - 2$$

Set $\frac{dy}{dx} = 0$:

$$4\cos^2 x + 2\cos x - 2 = 0 \Longrightarrow 2\cos^2 x + \cos x - 1 = 0$$

Factor: $(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = 1/2$: $x = \pi/3,\ 5\pi/3$. $\cos x = -1$: $x = \pi$.

Horizontal tangents at $x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$ in $[0,\ 2\pi]$.

■

Exercises

[★ routine ★★ intermediate ★★★ challenging]

7. [★] Differentiate: (a) $y = 3\cos(2x)$ (b) $y = \sin^3 x$ (c) $y = x^3 \tan x$ (d) $y = \sec(5x)$

8. [★] Find $\frac{dy}{dx}$: (a) $y = \sin(x^2 + 1)$ (b) $y = \cos^2(3x) - \sin^2(3x)$ (c) $y = x\cot x$

9. [★] Electric field $E = E_0\cos(\omega t - kx)$. Find $\frac{\partial E}{\partial t}$ and $\frac{\partial E}{\partial x}$. Interpret: what do these represent physically?

10. [★] Show $y = A\sin(nx) + B\cos(nx)$ satisfies $\frac{d^2y}{dx^2} + n^2 y = 0$. State what physical systems this governs.

11. [★★] Find all $x \in [0,\ 2\pi]$ where $y = 2\sin x + \sin(2x)$ has a horizontal tangent. Classify each as max, min, or inflection.

12. [★★] A searchlight 10 m from a wall rotates at $\frac{d\theta}{dt} = 2$ rad/s. Find the speed of the beam spot on the wall when $\theta = \pi/4$.

13. [★★★] Differentiate $y = \frac{\sin x}{x}$ for $x \neq 0$. Hence evaluate $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2}$.

4.2 Derivatives of Exponential Functions

Exponential functions model every process in which the rate of change is proportional to the current value. In electrical engineering: capacitor charging ($v = V_s\left(1 - e^{-t/RC}\right)$), inductor current decay ($i = I_0 e^{-Rt/L}$). In nuclear engineering: radioactive decay ($A = A_0 e^{-\lambda t}$). In heat transfer: Newton's cooling law ($T = T_\infty + (T_0 - T_\infty)e^{-ht}$). In control systems: step responses of first-order systems. The natural exponential e^x is uniquely the only non-trivial function that is its own derivative.

Figure 4.2.1: $y = e^x$. At any point the slope equals the y-value. At $x = 0$: slope $= 1$. At $x = 1$: slope $= e \approx 2.718$. This unique self-referential property — $\frac{d}{dx}(e^x) = e^x$ — makes exponentials the eigenfunctions of differentiation.

Figure 4.2.2: Comparing e^x, 2^x, 3^x. All pass through $(0,1)$. Slope at $x=0$ equals $\ln a$: $\ln 2 \approx 0.693$, $\ln e = 1$, $\ln 3 \approx 1.099$. Base e is unique: slope exactly equals value everywhere.

4.2.1 Derivation: $\frac{d}{dx}(e^x) = e^x$

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

The limit $\lim_{h\to 0}\frac{e^h-1}{h} = 1$ defines e: it is the unique positive real number for which this limit equals exactly 1. For any other base $a > 0$: $\lim_{h\to 0}\frac{a^h-1}{h} = \ln a$. This single fact is the origin of all logarithm properties.

Derivative of Natural Exponential

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

4.2.2 Derivation: $\frac{d}{dx}(a^x) = a^x \ln a$

Write a^x using $a = e^{\ln a}$, so $a^x = e^{x\ln a}$. Apply the Chain Rule with $u = x\ln a$:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\cdot \ln a = a^x \ln a$$

Derivative of General Exponential

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

When $a = e$: $\ln e = 1$, giving $\frac{d}{dx}(e^x) = e^x$. When $a = 2$: $\frac{d}{dx}(2^x) = 2^x \ln 2 \approx 0.693 \cdot 2^x$. When $a = 10$: $\frac{d}{dx}(10^x) = 10^x \ln 10 \approx 2.303 \cdot 10^x$.

4.2.3 Chain Rule Extension

$$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)}\cdot u'(x) \qquad \frac{d}{dx}\left[a^{u(x)}\right] = a^{u(x)}\cdot \ln a \cdot u'(x)$$

4.2.4 Worked Examples

Example 4.2.1

Differentiate: (a) $y = e^{3x^2-5}$ (b) $y = x^3 e^{-2x}$ (c) $y = e^{\sin x}$ (d) $y = 3^{2x+1}$

Solution:

14. $u = 3x^2 - 5$, $u' = 6x$:

$$\frac{dy}{dx} = 6x\, e^{3x^2-5}$$

15. Product Rule — $f = x^3$, $g = e^{-2x}$:

$$\frac{dy}{dx} = 3x^2 e^{-2x} + x^3(-2)e^{-2x} = x^2 e^{-2x}(3 - 2x)$$

16. $u = \sin x$, $u' = \cos x$:

$$\frac{dy}{dx} = e^{\sin x}\cos x$$

17. $\frac{d}{dx}(a^x) = a^x \ln a$, Chain Rule with $u = 2x + 1$, $a = 3$:

$$\frac{dy}{dx} = 3^{2x+1} \cdot \ln 3 \cdot 2 = 2\ln(3) \cdot 3^{2x+1}$$

■

Example 4.2.2

RC circuit: $v(t) = 12\left(1 - e^{-t/0.05}\right)$ V, $C = 200\ \mu$F. Find: (a) $i = C\, dv/dt$; (b) di/dt at $t = 0$; (c) time constant and steady-state voltage.

Solution:

18. $\frac{dv}{dt} = 12 \cdot \frac{1}{0.05} \cdot e^{-20t} = 240\, e^{-20t}$ V/s

$$i = 200 \times 10^{-6} \cdot 240\, e^{-20t} = 48\, e^{-20t}\ \text{mA}$$

19. $\frac{di}{dt} = 48 \times 10^{-3} \cdot (-20)\, e^{-20t}$. At $t = 0$: $\frac{di}{dt} = -0.96$ A/s.

20. Time constant $\tau = RC = 0.05$ s $= 50$ ms. Steady-state voltage $= 12$ V (as $t \to \infty$).

■

Example 4.2.3

Iodine-131: $T_{1/2} = 8.02$ days, $A(t) = A_0\, e^{-\lambda t}$. Find: (a) λ; (b) prove $\frac{dA}{dt} = -\lambda A$; (c) initial rate when $A_0 = 5 \times 10^6$ dis/s.

Solution:

21. $\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.6931}{8.02} = 0.08641\ \text{day}^{-1} = 1.000 \times 10^{-6}\ \text{s}^{-1}$

22. $\frac{dA}{dt} = A_0(-\lambda)e^{-\lambda t} = -\lambda\left(A_0 e^{-\lambda t}\right) = -\lambda A$ QED

23. $\left.\frac{dA}{dt}\right|_{t=0} = -\lambda A_0 = -1.000\times10^{-6}\cdot 5\times10^{6} = -5.00$ dis/s per second.

■

Example 4.2.4

Bacterial growth: $N(t) = N_0\,e^{kt}$. If N doubles in 3 hours, find k. Find $\frac{dN}{dt}$ at $t = 6$ h if $N_0 = 10^4$.

Solution:

Doubling: $N_0\,e^{3k} = 2N_0 \Longrightarrow e^{3k} = 2 \Longrightarrow k = \frac{\ln 2}{3} = 0.2310\ \text{h}^{-1}$.

$$\frac{dN}{dt} = N_0\,k\,e^{kt}. \quad \text{At } t=6: \quad N(6) = 10^4\,e^{0.2310\times6} = 10^4\,e^{1.386} = 10^4\cdot 4 = 40\,000$$

$$\left.\frac{dN}{dt}\right|_{t=6} = k\,N(6) = 0.2310\times 40\,000 = 9240 \text{ bacteria/hour}$$

■

Exercises

24. [★] Differentiate: (a) $y = e^{4x-7}$ (b) $y = e^{x^2}$ (c) $y = x\,e^{x}$ (d) $y = e^{-x}\cos x$

25. [★] Find $\frac{dy}{dx}$: (a) $y = 5^{x}$ (b) $y = 2^{x^2}$ (c) $y = 10^{\sin x}$

26. [★] Find $\frac{d^2y}{dx^2}$ for $y = e^{ax}$ and verify $y'' - a^2y = 0$.

27. [★★] Capstan equation: $\frac{dT}{dx} = \mu T$. Verify $T = T_0\,e^{\mu x}$ satisfies this. With $T_0 = 100$ N, $\mu = 0.3$, find $\frac{dT}{dx}$ at $x = \pi/4$.

28. [★★] Find the coordinates of the maximum of $y = x\,e^{-x}$ and the tangent at $x = 2$.

29. [★★★] Prove $y = e^{ax}(A\cos bx + B\sin bx)$ satisfies $y'' - 2ay' + (a^2 + b^2)y = 0$.

4.3 Derivatives of Logarithmic Functions

The natural logarithm $\ln x$ is the inverse of e^{x} and has the simplest derivative of any transcendental function: $1/x$. This result appears in entropy formulas ($H = -k\sum p\ln p$), decibel calculations ($L = 20\log_{10}(P/P_0)$), information theory (Shannon entropy), the integration of rational fractions, and the chain of calculations underpinning logarithmic differentiation in Section 4.7.

Figure 4.3.1: $y = \ln x$ (blue) and its derivative $y = 1/x$ (gold dashed). At $x = 1$: $\ln 1 = 0$ and slope $= 1$. As $x \to 0^+$: $\ln \to -\infty$. As $x \to \infty$: $\ln \to +\infty$ but slowly. Domain: $x > 0$ only.

4.3.1 Derivation: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Let $y = \ln x$, so $e^y = x$. Differentiate both sides implicitly with respect to x:

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Derivative of Natural Logarithm

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \qquad (x > 0)$$

For negative x: $\frac{d}{dx}(\ln|x|) = \frac{1}{x}(x \neq 0)$. This form is essential when integrating $1/x$ over any interval.

4.3.2 Derivation: $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Use the change-of-base formula $\log_a x = \frac{\ln x}{\ln a}$, with $\ln a$ a constant:

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

Derivative of $\log_a$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

For $a = 10$ (common log): $\frac{d}{dx}(\log_{10} x) = \frac{1}{x \ln 10} \approx \frac{0.4343}{x}$. For $a = 2$ (binary log): $\frac{d}{dx}(\log_2 x) = \frac{1}{x \ln 2} \approx \frac{1.4427}{x}$.

4.3.3 Chain Rule Extension

$$\frac{d}{dx}\left[\ln\left(u(x)\right)\right] = \frac{u'(x)}{u(x)} \qquad \frac{d}{dx}\left[\log_a\left(u(x)\right)\right] = \frac{u'(x)}{u(x) \ln a}$$

The result u'/u is the *logarithmic derivative* — the foundation of logarithmic differentiation (Section 4.7).

4.3.4 Worked Examples

Example 4.3.1

Differentiate: (a) $y = \ln(3x^2 + 5)$ (b) $y = x^2 \ln x$ (c) $y = \ln(\sin x)$ (d) $y = \log_{10}(x^2 - 1)$

Solution:

30. $u = 3x^2 + 5, u' = 6x$: $\frac{dy}{dx} = \frac{6x}{3x^2+5}$

31. Product Rule:

$$\frac{dy}{dx} = 2x\ln x + x^2 \cdot \frac{1}{x} = 2x\ln x + x = x(2\ln x + 1)$$

32. $u = \sin x, u' = \cos x$: $\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

33. $u = x^2 - 1, u' = 2x$, base $a = 10$:

$$\frac{dy}{dx} = \frac{2x}{(x^2 - 1)\ln 10}$$

■

Example 4.3.2

Sound level $L = 20\log_{10}(P/P_0)$ dB, $P_0 = 20\ \mu$Pa. (a) Find $\frac{dL}{dP}$. (b) At $P = 1$ Pa how many dB per Pascal? (c) Find $\frac{d^2L}{dP^2}$ and interpret.

Solution:

34. $\frac{dL}{dP} = \frac{20}{P\ln 10} = \frac{8.686}{P}$ dB/Pa

35. At $P = 1$ Pa: $\frac{dL}{dP} = 8.686$ dB/Pa. Each additional Pascal raises the level by 8.69 dB.

36. $\frac{d^2L}{dP^2} = -\frac{8.686}{P^2} < 0$ for all $P > 0$. Concave down: equal absolute pressure increases give diminishing dB gains as P rises. This is why the dB scale compresses the vast 10^{12}:1 range of human hearing into a manageable 120 dB scale.

■

Example 4.3.3

Show $\frac{d}{dx}[\ln|\sec x + \tan x|] = \sec x$, and write down the antiderivative of $\sec x$.

Solution:

$u = \sec x + \tan x$. $u' = \sec x\tan x + \sec^2 x = \sec x(\tan x + \sec x)$.

$$\frac{dy}{dx} = \frac{u'}{u} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

Therefore: $\int \sec x\, dx = \ln|\sec x + \tan x| + C$. This antiderivative is impossible to guess by inspection; differentiation of the answer verifies it and reads the result in reverse.

■

Example 4.3.4

$\text{pH} = -\log_{10}[\text{H}^+]$. Find $\frac{d(\text{pH})}{d[\text{H}^+]}$ and the sensitivity of pH to hydrogen ion concentration at $[\text{H}^+] = 10^{-7}$ mol/L (neutral water).

Solution:

$$\frac{d(\text{pH})}{d[\text{H}^+]} = \frac{-1}{[\text{H}^+]\ln 10} = \frac{-0.4343}{[\text{H}^+]}$$

At $[\text{H}^+] = 10^{-7}$: $\frac{d(\text{pH})}{d[\text{H}^+]} = \frac{-0.4343}{10^{-7}} = -4.343 \times 10^6$ pH units per (mol/L).

The enormous magnitude means pH is extremely sensitive to $[\text{H}^+]$ changes near neutrality. A change of only 10^{-8} mol/L in $[\text{H}^+]$ shifts pH by 0.0434 units — detectable by a pH meter.

■

Exercises

37. [★] Differentiate: (a) $y = \ln(x^3 - 2x)$ (b) $y = \sqrt{\ln x}$ (c) $y = \ln\left(x + \sqrt{x^2+1}\right)$ (d) $y = \ln(\cos x)$

38. [★] Find $\frac{dy}{dx}$: (a) $y = x\ln x - x$ (b) $y = (\ln x)^2$ (c) $y = \ln\frac{x}{x^2+1}$

39. [★★] Show $\frac{d}{dx}[\ln(\sin x)] = \cot x$ from first principles (definition of derivative).

40. [★★] Entropy $H(p) = -p\ln p - (1-p)\ln(1-p)$. Find $\frac{dH}{dp}$ and the value of p that maximises H. Interpret.

41. [★★] Find $\frac{d^2y}{dx^2}$ for $y = x^2\ln x$.

42. [★★★] Filter gain $G(f) = \frac{1}{\sqrt{1+(f/f_0)^2}}$. Write $G_{\text{dB}} = 20\log_{10}G$. Find $\frac{dG_{\text{dB}}}{df}$ and evaluate at $f = f_0$ (the -3 dB point).

4.4 Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions convert known ratios back into angles — essential in robotics (joint angles from position coordinates), AC circuit analysis (phase angles from real and imaginary impedance components), navigation (bearing calculations), and optics

(angle of refraction). In calculus, they arise as the antiderivatives of algebraic expressions involving $\sqrt{1-x^2}$ and $1/(1+x^2)$, which appear throughout integration of rational functions and solution of differential equations.

Figure 4.4.1: $\arctan x$ *(blue) with horizontal asymptotes at* $\pm\pi/2$*, and its derivative* $1/(1+x^2)$ *(gold) — a bell curve with maximum 1 at* $x=0$ *decaying to 0 as* $|x|\to\infty$.

Figure 4.4.2: Unit circle geometric derivation. Setting $\sin y = x$ *and differentiating implicitly gives* $dy/dx = 1/\cos y = 1/\sqrt{1-x^2}$*. Setting* $\tan y = x$ *gives* $dy/dx = 1/\sec^2 y = 1/(1+x^2)$.

4.4.1 Derivation: $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Let $y = \arcsin x$, so $\sin y = x$ with $-\pi/2 \le y \le \pi/2$ (principal range). Differentiate both sides implicitly with respect to x:

$$\cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

From $\sin y = x$: $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. Positive root since $\cos y \ge 0$ on the principal range.

Derivative of arcsin

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}} \qquad (-1 < x < 1)$$

4.4.2 Derivation: $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

Let $y = \arccos x$, so $\cos y = x$ with $0 \le y \le \pi$. Differentiating implicitly:

$$-\sin y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$$

Derivative of arccos

$$\boxed{\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}} \qquad (-1 < x < 1)$$

Note: $\frac{d}{dx}(\arcsin x) + \frac{d}{dx}(\arccos x) = 0$, reflecting the identity $\arcsin x + \arccos x = \pi/2$ (their sum is constant).

4.4.3 Derivation: $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Let $y = \arctan x$, so $\tan y = x$ with $-\pi/2 < y < \pi/2$. Differentiating:

$$\sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

Using $\sec^2 y = 1 + \tan^2 y = 1 + x^2$: $\cos^2 y = \frac{1}{1+x^2}$.

Derivative of arctan

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}} \qquad (\text{all } x)$$

4.4.4 Remaining Inverse Trig Derivatives

$$\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2} \qquad (\text{all } x)$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

Chain Rule forms used in practice:

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

4.4.5 Worked Examples

Example 4.4.1

Differentiate: (a) $y = \arcsin(3x)$ (b) $y = \arctan(x^2)$ (c) $y = x\arctan x - 1/2\ln(1+x^2)$ (d) $y = \arcsin(\sqrt{x})$

Solution:

43. $u = 3x, u' = 3$:

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-9x^2}}$$

44. $u = x^2, u' = 2x$:

$$\frac{dy}{dx} = \frac{2x}{1+x^4}$$

45. Product Rule + Chain: $\frac{d}{dx}[x\arctan x] = \arctan x + \frac{x}{1+x^2}$. $\frac{d}{dx}[-1/2\ln(1+x^2)] = -\frac{x}{1+x^2}$:

$$\frac{dy}{dx} = \arctan x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \arctan x$$

46. $u = \sqrt{x} = x^{1/2}$, $u' = \frac{1}{2\sqrt{x}}$:

$$\frac{dy}{dx} = \frac{1/(2\sqrt{x})}{\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

■

Example 4.4.2

Robot joint angle: $\theta(x) = \arctan(x/h)$, $h = 0.4$ m. Find $d\theta/dx$ at $x = 0.3$ m and interpret.

Solution:

$u = x/0.4$, $u' = 2.5$:

$$\frac{d\theta}{dx} = \frac{2.5}{1+(x/0.4)^2} = \frac{2.5}{1+6.25x^2} \text{ rad/m}$$

At $x = 0.3$: $\frac{d\theta}{dx} = \frac{2.5}{1+0.5625} = \frac{2.5}{1.5625} = 1.6$ rad/m.

Moving the gripper 1 mm upward rotates the joint 1.6 milliradians. This angular sensitivity must be within the servo resolution for precise positioning.

■

Example 4.4.3

Prove $\frac{d}{dx}[\arctan x + \arctan(1/x)] = 0$ for $x > 0$, and find the constant value.

Solution:

$\frac{d}{dx}[\arctan(1/x)]$: $u = 1/x$, $u' = -1/x^2$:

$$\frac{d}{dx}[\arctan(1/x)] = \frac{-1/x^2}{1+1/x^2} = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\arctan x + \arctan(1/x)] = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Since the derivative is 0 for all $x > 0$, the function is constant. At $x = 1$: $\pi/4 + \pi/4 = \pi/2$.

$$\arctan x + \arctan(1/x) = \frac{\pi}{2} \qquad \text{for all } x > 0$$

■

Exercises

47. [★] Differentiate: (a) $y = \arcsin(x/2)$ (b) $y = \arccos(1 - 2x^2)$ (c) $y = \arctan\left(\sqrt{x}\right)$ (d) $y = \operatorname{arccot}(e^x)$

48. [★★] Find $\frac{dy}{dx}$: (a) $y = x^2 \arctan x$ (b) $y = \frac{\arcsin x}{\sqrt{1-x^2}}$ (c) $y = \arctan\frac{1-x}{1+x}$

49. [★★] Observer 50 m from a wall; ball rises at 8 m/s. Find $d\theta/dt$ when $h = 30$ m.

50. [★★] Find $\frac{d^2y}{dx^2}$ for $y = \arctan x$.

51. [★★★] Derive $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$ for $x > 1$ using implicit differentiation on $\sec y = x$.

52. [★★] Linearise $\arcsin x$ about $x = 0$. Estimate $\arcsin(0.1)$ in radians and find the percentage error.

4.5 Derivatives of Hyperbolic Functions

Hyperbolic functions are defined using the natural exponential and arise wherever physical problems combine exponential growth and decay. They describe: the shape of hanging cables and chains (catenary), the temperature distribution in extended surfaces (fins), voltage and current profiles along transmission lines (telegraph equations), and the solutions of second-order differential equations with positive constant coefficients. Their derivatives parallel trigonometric derivatives almost exactly, with the crucial sign difference that $\frac{d}{dx}(\cosh x) = +\sinh x$ rather than $-\cosh x$.

Figure 4.5.1: $\sinh x$ *(red),* $\cosh x$ *(blue),* $\tanh x$ *(green dashed).* $\cosh x \geq 1$ *always, with minimum at* $x = 0$. $\tanh x$ *is bounded between* -1 *and* $+1$. *Note:* $\frac{d}{dx}(\cosh x) = +\sinh x$*, not* $-\sinh x$.

Figure 4.5.2: Catenary $y = a\cosh(x/a)$*: the natural shape of a uniform flexible cable under gravity. The slope* $\sinh(x/a)$ *and arc element* $\cosh(x/a)\,dx$ *follow directly from differentiation.*

4.5.1 Definitions and the Fundamental Identity

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Fundamental Identity

$$\cosh^2 x - \sinh^2 x = 1$$

Compare with $\cos^2 x + \sin^2 x = 1$: the minus sign on the left arises from the e^{-x} terms having the same (not opposite) sign in cosh and sinh. This sign difference has profound consequences: the hyperbola $x^2 - y^2 = 1$ (parameterised by cosh and sinh) has very different geometry from the circle $x^2 + y^2 = 1$.

4.5.2 Derivation: $\frac{d}{dx}(\sinh x) = \cosh x$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

Derivative of sinh

$$\boxed{\frac{d}{dx}(\sinh x) = \cosh x}$$

4.5.3 Derivation: $\frac{d}{dx}(\cosh x) = \sinh x$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

Derivative of cosh

$$\boxed{\frac{d}{dx}(\cosh x) = \sinh x}$$

Critical distinction: $\frac{d}{dx}(\cos x) = -\sin x$, but $\frac{d}{dx}(\cosh x) = +\sinh x$. No minus sign. This means $\cosh x$ has a minimum (not a maximum) at $x = 0$, which is why a hanging cable curves upward.

4.5.4 Derivation: $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Quotient Rule on $\tanh x = \sinh x / \cosh x$:

$$\frac{d}{dx}(\tanh x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Derivative of tanh

$$\boxed{\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x}$$

4.5.5 Complete Reference Table

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$1/\sqrt{x^2+1}$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$1/\sqrt{x^2-1}, x > 1$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$1/(1-x^2), \lvert x\rvert < 1$

4.5.6 Worked Examples

Example 4.5.1

Differentiate: (a) $y = \sinh(4x)$ (b) $y = \cosh^2 x$ (c) $y = \tanh\left(\sqrt{x}\right)$ (d) $y = x\sinh x - \cosh x$

Solution:

53. Chain Rule: $\frac{d}{dx}[\sinh(4x)] = \cosh(4x) \cdot 4 = 4\cosh(4x)$

54. Chain Rule: $\frac{d}{dx}[\cosh^2 x] = 2\cosh x \sinh x = \sinh(2x)$

55. $u = \sqrt{x}, u' = 1/\left(2\sqrt{x}\right)$:

$$\frac{dy}{dx} = \frac{\operatorname{sech}^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

56. Product Rule on $x\sinh x$, then subtract $\sinh x$:

$$\frac{dy}{dx} = \sinh x + x\cosh x - \sinh x = x\cosh x$$

■

Example 4.5.2

Catenary $y(x) = a\cosh(x/a)$. Find: (a) slope; (b) arc length element ds; (c) curvature κ.

Solution:

57. $\frac{dy}{dx} = a \cdot \sinh(x/a) \cdot (1/a) = \sinh(x/a)$

58. $ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + \sinh^2(x/a)}\, dx = \sqrt{\cosh^2(x/a)}\, dx = \cosh(x/a)\, dx$

Arc length $= \int \cosh(x/a)\, dx = a\sinh(x/a)$. An elegant exact result.

59. $\frac{d^2y}{dx^2} = \frac{1}{a}\cosh(x/a)$. Denominator $= \cosh^3(x/a)$.

$$\kappa = \frac{(1/a)\cosh(x/a)}{\cosh^3(x/a)} = \frac{1}{a\cosh^2(x/a)} = \frac{\operatorname{sech}^2(x/a)}{a}$$

Maximum curvature $\kappa = 1/a$ at $x = 0$ (lowest point). Cable straightens towards the supports.

■

Example 4.5.3

Transmission line voltage $V(x) = V_0\cosh(\gamma x)$. Find $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$ and verify the telegrapher's equation $\frac{d^2V}{dx^2} = \gamma^2 V$.

Solution:

$$\frac{dV}{dx} = V_0\gamma\sinh(\gamma x)$$

$$\frac{d^2V}{dx^2} = V_0\gamma^2\cosh(\gamma x) = \gamma^2 V(x)$$

$$\frac{d^2V}{dx^2} = \gamma^2 V \qquad \text{QED}$$

This is the telegrapher's equation for a lossless transmission line. The hyperbolic solution is why transmission line analysis uses sinh and cosh throughout RF and microwave engineering.

■

Exercises

60. [★] Differentiate: (a) $y = \cosh(3x^2)$ (b) $y = \sinh x\cosh x$ (c) $y = \tanh(e^x)$ (d) $y = \operatorname{sech}^2 x$

61. [★] Verify $y = A\cosh(nx) + B\sinh(nx)$ satisfies $y'' = n^2 y$.

62. [★★] Fin temperature: $T(x) = T_b \dfrac{\cosh\big(m(L-x)\big)}{\cosh(mL)}$. Find $\dfrac{dT}{dx}$ and evaluate at $x = 0$.

63. [★★★] Prove $\tanh^{-1}x = 1/2 \ln\dfrac{1+x}{1-x}$ for $|x| < 1$. Differentiate both sides to verify $\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1-x^2}$.

64. [★★★] Show that $\dfrac{d}{dx}\left[\dfrac{x\sqrt{1+x^2}}{2} + \dfrac{\sinh^{-1}x}{2}\right] = \sqrt{1+x^2}$.

4.6 Implicit Differentiation

When a relationship between x and y cannot be (or need not be) solved explicitly as $y = f(x)$, implicit differentiation provides dy/dx directly from the equation $F(x, y) = 0$. This situation arises constantly in engineering: the equation of a circle or ellipse, the van der Waals gas equation, gear contact equations, stress-strain compatibility relations, and the Clausius-Clapeyron thermodynamic relation all define y implicitly. The technique is also essential for deriving the derivatives of inverse functions (as used in Sections 4.4 and 3.10).

Figure 4.6.1: Implicit differentiation of $x^2 + y^2 = r^2$. The tangent (red) has slope $-x/y$ and is perpendicular to the radius OP. The normal (green) passes through the centre O. Both follow from $dy/dx = -x/y$.

4.6.1 The Four-Step Method

Step 1: Differentiate every term in $F(x, y) = 0$ with respect to x.

Step 2: For every term containing y, apply the Chain Rule: $\dfrac{d}{dx}[f(y)] = f'(y) \cdot \dfrac{dy}{dx}$.

Step 3: Collect all terms containing $\dfrac{dy}{dx}$ on the left side.

Step 4: Factor out $\dfrac{dy}{dx}$ and divide to solve.

The result gives $\dfrac{dy}{dx}$ in terms of both x and y. To find the slope at a specific point (x_0, y_0), substitute both coordinates. For $\dfrac{d^2y}{dx^2}$: differentiate $\dfrac{dy}{dx}$ again using the same implicit technique.

4.6.2 Worked Examples

Example 4.6.1

Find $\dfrac{dy}{dx}$ for ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find tangent and normal at $P = \left(\dfrac{a}{\sqrt{2}}, \dfrac{b}{\sqrt{2}}\right)$.

Solution:

Differentiate with respect to x:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

At $P = \left(a/\sqrt{2},\, b/\sqrt{2}\right)$: $\frac{dy}{dx} = -\frac{b^2(a/\sqrt{2})}{a^2(b/\sqrt{2})} = -\frac{b}{a}$

Tangent: $y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right) \Rightarrow bx + ay = ab\sqrt{2}$

Normal slope $= a/b$. Normal: $y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$

■

Example 4.6.2

Find $\frac{dy}{dx}$ for $x^3 + y^3 - 3xy = 7$ at the point (2,1).

Solution:

Differentiate term by term:

$$3x^2 + 3y^2\frac{dy}{dx} - 3\left[y + x\frac{dy}{dx}\right] = 0$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

At (2,1): $\frac{dy}{dx} = \frac{1-4}{1-2} = \frac{-3}{-1} = 3$

■

Example 4.6.3

Clausius-Clapeyron: $\frac{dP}{dT} = \frac{\Delta H_v\, P}{R\,T^2}$. Find $\frac{dT}{dP}$. Evaluate at $T = 373$ K, $P = 101\,325$ Pa, $\Delta H_v/R = 5120$ K.

Solution:

$$\frac{dT}{dP} = \frac{1}{dP/dT} = \frac{R\,T^2}{\Delta H_v\, P}$$

At $T = 373$, $P = 101\,325$:

$$\frac{dT}{dP} = \frac{R\cdot 373^2}{5120\cdot R\cdot 101\,325} = \frac{139\,129}{5.188\times 10^{8}} = 2.68\times 10^{-4}\ \text{K/Pa}$$

Increasing pressure by 1 kPa raises the boiling point by 0.0268°C. A pressure cooker at 120 kPa boils water at approximately 104°C.

■

Example 4.6.4

Find $\frac{d^2y}{dx^2}$ for circle $x^2 + y^2 = r^2$.

Solution:

First derivative: $2x + 2y\,\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$

Differentiate again (Quotient Rule, treating y as a function of x):

$$\frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x \cdot (dy/dx)}{y^2} = -\frac{y + x^2/y}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{r^2}{y^3}$$

For $y > 0$: $\frac{d^2y}{dx^2} < 0 \Longrightarrow$ upper semicircle is concave downward. Geometrically correct.

■

Example 4.6.5

Van der Waals gas: $(P + a/V^2)(V - b) = RT$. Find $\frac{dV}{dT}$ at constant P and interpret physically.

Solution:

Differentiate both sides with respect to T at constant P:

$$\left(P + \frac{a}{V^2}\right)\frac{dV}{dT} + (V - b)\left(-\frac{2a}{V^3}\right)\frac{dV}{dT} = R$$

$$\frac{dV}{dT}\left[\left(P + \frac{a}{V^2}\right) - \frac{2a(V - b)}{V^3}\right] = R$$

$$\frac{dV}{dT} = \frac{R}{\left(P + \frac{a}{V^2}\right) - \frac{2a(V - b)}{V^3}}$$

This is the thermal expansivity of a van der Waals gas. For an ideal gas ($a = b = 0$) it reduces to $\frac{dV}{dT} = R/P$, giving $\alpha = R/(PV) = 1/T$.

■

Exercises

65. [★] Find $\frac{dy}{dx}$: (a) $x^2 - xy + y^2 = 7$ (b) $\sin(xy) = y$ (c) $e^{xy} = x + y$

66. [★★] Find the gradient of $x^4 + y^4 = 16$ at the point where $x = y > 0$. Find the tangent equation.

67. [★★] For the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, show $\frac{dy}{dx} = -(y/x)^{1/3}$.

68. [★] Ideal gas $PV = nRT$. Find $\frac{dP}{dV}$ at constant T and $\frac{dP}{dT}$ at constant V. Interpret as Boyle's and Gay-Lussac's laws.

69. [★★★] Find $\frac{d^2y}{dx^2}$ for hyperbola $x^2 - y^2 = 1$ and verify $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$.

70. [★★] Re-derive $\frac{d}{dx}(\arctan x)$ using implicit differentiation of $\tan y = x$.

4.7 Logarithmic Differentiation

Three classes of functions resist standard differentiation rules but yield immediately when logarithms are applied: (1) large products or quotients of many factors, where logarithms convert multiplication into addition; (2) functions with a variable exponent $f(x)^{g(x)}$ — neither the Power Rule (constant exponent) nor the Exponential Rule (constant base) applies; and (3) expressions where the natural logarithm itself reveals a simpler structure. The technique uses ln laws to convert difficult multiplication into simpler addition, then differentiates implicitly.

Figure 4.7.1: $y = x^x$: minimum at $x = 1/e \approx 0.368$ where $\frac{dy}{dx} = x^x(\ln x + 1) = 0$. This function cannot be differentiated by either the Power Rule or the Exponential Rule alone — logarithmic differentiation is required.

4.7.1 The Method

Step 1: Take the natural logarithm of both sides: $\ln y = \ln[f(x)]$.

Step 2: Apply log laws: $\ln(AB) = \ln A + \ln B$; $\ln(A/B) = \ln A - \ln B$; $\ln(A^n) = n\ln A$.

Step 3: Differentiate both sides with respect to x (left side: $\frac{1}{y}\,y'$).

Step 4: Multiply both sides by y to obtain $y' = \frac{dy}{dx}$.

4.7.2 Worked Examples

Example 4.7.1

Differentiate $y = x^x$ for $x > 0$.

Solution:

$\ln y = x\ln x$. Differentiate (Product Rule on right):

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Setting $\frac{dy}{dx} = 0$: $\ln x + 1 = 0 \Longrightarrow x = e^{-1} = 1/e \approx 0.368$ (minimum, confirmed by second derivative test).

■

Example 4.7.2

Differentiate $y = \sqrt{\frac{(x+1)^2(2x-1)^3}{x^2+4}}$.

Solution:

$$\ln y = \frac{1}{2}[2\ln(x+1) + 3\ln(2x-1) - \ln(x^2+4)]$$

Differentiate:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{x+1} + \frac{6}{2x-1} - \frac{2x}{x^2+4}\right]$$

$$\frac{dy}{dx} = y\cdot\frac{1}{2}\left[\frac{2}{x+1} + \frac{6}{2x-1} - \frac{2x}{x^2+4}\right]$$

■

Example 4.7.3

Differentiate $y = (\sin x)^{\cos x}$.

Solution:

$\ln y = \cos x\ln(\sin x)$. Differentiate (Product Rule on right):

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\sin x\ln(\sin x) + \cos x\cdot\frac{\cos x}{\sin x} = -\sin x\ln(\sin x) + \frac{\cos^2 x}{\sin x}$$

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[-\sin x\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$$

■

Example 4.7.4

Arrhenius equation: $k = A\,e^{-E_a/(RT)}$. Use logarithmic differentiation to find dk/dT. Show $\frac{d(\ln k)}{dT} = \frac{E_a}{RT^2}$.

Solution:

$\ln k = \ln A - \frac{E_a}{RT}$. Differentiate with respect to T:

$$\frac{1}{k}\cdot\frac{dk}{dT} = \frac{E_a}{RT^2} \Longrightarrow \frac{dk}{dT} = \frac{k\,E_a}{RT^2}$$

This shows the rate constant increases sharply with temperature for large E_a. The rule "rate doubles per 10°C" corresponds to $E_a \approx 52–60$ kJ/mol for most chemical reactions.

■

Example 4.7.5

Drag force $F = 1/2\,C_D\rho A v^2$. Show $\frac{1}{F}\frac{dF}{dv} = \frac{2}{v}$ using logarithmic differentiation. Interpret.

Solution:

$\ln F = \ln(C_D\rho A/2) + 2\ln v$ (treating constants collectively). Differentiate:

$$\frac{1}{F}\cdot\frac{dF}{dv} = \frac{2}{v}$$

Interpretation: a 1% increase in speed produces a 2% increase in drag force. A 10% speed increase gives a 21% drag increase — which is why aerodynamic drag is so costly at high speeds.

■

Exercises

71. [★★] Differentiate: (a) $y = x^{\sin x}$ (b) $y = (\ln x)^x$ (c) $y = (\tan x)^{e^x}$

72. [★] Differentiate $y = \sqrt{(x-1)(x-2)(x-3)}$ for $x > 3$.

73. [★★] Find $\frac{dy}{dx}$ for $y = (1 + 1/x)^x$. Show that as $x \to \infty$, $y \to e$.

74. [★★★] Find and classify the stationary point of $y = x^{1/x}$ for $x > 0$.

75. [★★] Power dissipated $P = V^2/R$. Show $\frac{1}{P}\frac{dP}{dV} = \frac{2}{V}$ and $\frac{1}{P}\frac{dP}{dR} = -\frac{1}{R}$. Interpret.

4.8 Parametric Differentiation

Many curves in engineering and physics are most naturally described by a pair of parametric equations $x = x(t)$ and $y = y(t)$ in terms of a parameter t. The path of a projectile (t = time), gear tooth profiles (t = rolling angle), cycloid paths of pistons in Wankel engines, elliptical orbits (t = eccentric anomaly), and Lissajous figures all resist description as $y = f(x)$ but are immediately accessible parametrically. The technique of parametric differentiation yields dy/dx and d^2y/dx^2 directly without eliminating the parameter.

Figure 4.8.1: Cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$: path traced by a point on a rolling circle. Cusps at $\theta = 2n\pi$ (where slope is vertical), horizontal tangents at tops ($\theta = (2n+1)\pi$). Slope $= \cot(\theta/2)$.

Figure 4.8.2: Parametric motion: velocity vector (red, tangent to path), acceleration vector (green). Speed $= |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}$. $\frac{dy}{dx} = \dot{y}/\dot{x}$. $\frac{d^2y}{dx^2} = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]/\dot{x}$.

4.8.1 First Derivative

By the Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. Solving for $\frac{dy}{dx}$ when $\frac{dx}{dt} \neq 0$:

Parametric First Derivative

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Horizontal tangents: $\frac{dy}{dt} = 0$ with $\frac{dx}{dt} \neq 0$. Vertical tangents: $\frac{dx}{dt} = 0$ with $\frac{dy}{dt} \neq 0$. Cusp: both $= 0$ (examine the limit of dy/dx).

4.8.2 Second Derivative

Differentiate dy/dx with respect to x via the Chain Rule:

Parametric Second Derivative

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

CRITICAL WARNING: $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$. This is the most common error in parametric problems. Always differentiate dy/dx with respect to t, then divide by dx/dt.

4.8.3 Worked Examples

Example 4.8.1

Curve: $x = t^2 - 3$, $y = t^3 - 3t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Find horizontal tangent points and their equations.

Solution:

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$

$$\frac{dy}{dx} = \frac{3(t^2-1)}{2t}$$

For $\frac{d^2y}{dx^2}$: differentiate $\frac{dy}{dx}$ with respect to t:

$$\frac{d}{dt}\left[\frac{3(t^2-1)}{2t}\right] = \frac{6t(2t) - 3(t^2-1)\cdot 2}{4t^2} = \frac{6t^2+6}{4t^2}$$

$$\frac{d^2y}{dx^2} = \frac{(6t^2+6)/(4t^2)}{2t} = \frac{3(t^2+1)}{4t^3}$$

Horizontal tangents: $\frac{dy}{dt} = 0 \Longrightarrow t = \pm 1$.

$t = +1$: $x = -2$, $y = -2$. Tangent: $y = -2$. $\quad$ $t = -1$: $x = -2$, $y = 2$. Tangent: $y = 2$.

■

Example 4.8.2

Cycloid: $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$. Find $\frac{dy}{dx}$. Tangent at $\theta = \pi/2$. Prove cusps at $\theta = 2n\pi$.

Solution:

$\frac{dx}{d\theta} = r(1-\cos\theta)$, $\frac{dy}{d\theta} = r\sin\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \cot\left(\frac{\theta}{2}\right) \qquad \text{[by half-angle identities]}$$

At $\theta = \pi/2$: $\frac{dy}{dx} = \cot(\pi/4) = 1$. Point: $x = r(\pi/2 - 1)$, $y = r$.

Tangent: $Y - r = 1 \cdot \big(X - r(\pi/2-1)\big) \Longrightarrow Y = X + r(2 - \pi/2)$

At $\theta = 0$: $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$. Examining limit: $\lim_{\theta\to 0}\cot(\theta/2) = +\infty \Longrightarrow$ vertical tangent (cusp). The rolling circle momentarily stops at the contact point.

■

Example 4.8.3

Circular motion: $x = R\cos(\omega t)$, $y = R\sin(\omega t)$. Find velocity vector, speed, and acceleration. Prove centripetal.

Solution:

Velocity: $\dot{x} = -R\omega\sin(\omega t)$, $\dot{y} = R\omega\cos(\omega t)$

Speed $= \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{R^2\omega^2(\sin^2 + \cos^2)} = R\omega$ (constant)

Acceleration: $\ddot{x} = -R\omega^2\cos(\omega t) = -\omega^2 x$

$$\ddot{y} = -R\omega^2\sin(\omega t) = -\omega^2 y$$

Acceleration vector $= -\omega^2(x, y)$: directed towards origin (centre). Magnitude $R\omega^2 = v^2/R$. Centripetal. QED.

■

Example 4.8.4

Ellipse: $x = 3\cos t$, $y = 2\sin t$. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$. Find the tangent and normal at $t = \pi/4$. Determine concavity.

Solution:

$\frac{dx}{dt} = -3\sin t, \frac{dy}{dt} = 2\cos t$

$$\frac{dy}{dx} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2}{3}\csc^2 t$$

$$\frac{d^2y}{dx^2} = \frac{(2/3)\csc^2 t}{-3\sin t} = -\frac{2}{9\sin^3 t}$$

At $t = \pi/4$: $\frac{dy}{dx} = -\frac{2}{3}$. $x = \frac{3}{\sqrt{2}}, y = \sqrt{2}$.

Tangent: $y - \sqrt{2} = -\frac{2}{3}\left(x - \frac{3}{\sqrt{2}}\right) \Rightarrow 2x + 3y = \frac{3}{\sqrt{2}} \cdot 2 + \frac{3}{\sqrt{2}} = \frac{9}{\sqrt{2}} = \frac{9\sqrt{2}}{2}$

$\frac{d^2y}{dx^2} = -\frac{2}{9(1/\sqrt{2})^3} = -\frac{4\sqrt{2}}{9} < 0 \Rightarrow$ concave down.

■

Exercises

76. [★] Find $\frac{dy}{dx}$: (a) $x = t^3,\ y = t^2$ (b) $x = 1/t,\ y = t + 1/t$ (c) $x = e^t,\ y = e^{-t}$

77. [★★] Ellipse $x = 3\cos t$, $y = 2\sin t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Find tangent at $t = \pi/4$.

78. [★★] For $x = t - \sin t$, $y = 1 - \cos t$, find $\frac{d^2y}{dx^2}$ and determine where curve is concave up.

79. [★★] Particle: $x(t) = t^2 - 4$, $y(t) = t^3 - 3t$. Find all times when the tangent is vertical.

80. [★★] Involute: $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$. Show $\frac{dy}{dx} = \tan t$.

81. [★★★] Parabola $x = at^2$, $y = 2at$. Find $\frac{d^2y}{dx^2}$ and show curvature at vertex is $1/a$.

Chapter 4 — Complete Derivative Reference Table

All differentiation rules from Chapters 3 and 4. For composite functions apply the Chain Rule: multiply by du/dx whenever the argument is $u(x)$ rather than x.

Basic and Power Rules

Rule	Formula
Constant	$\frac{d}{dx}(c) = 0$
Power	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant multiple	$\frac{d}{dx}[c\,f] = c\,f'$
Sum/Difference	$\frac{d}{dx}[f \pm g] = f' \pm g'$
Product	$\frac{d}{dx}[f \cdot g] = f'g + fg'$
Quotient	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g) \cdot g'$

Trigonometric

$f(x)$	$f'(x)$
$\sin x$	$\cos x$

$f(x)$	$f'(x)$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Exponential and Logarithmic

$f(x)$	$f'(x)$
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$1/x$
$\log_a x$	$1/(x \ln a)$

Inverse Trigonometric

$f(x)$	$f'(x)$
$\arcsin x$	$1/\sqrt{1-x^2}$
$\arccos x$	$-1/\sqrt{1-x^2}$
$\arctan x$	$1/(1+x^2)$
$\operatorname{arccot} x$	$-1/(1+x^2)$
$\operatorname{arcsec} x$	$1/\left(\lvert x\rvert\sqrt{x^2-1}\right)$
$\operatorname{arccsc} x$	$-1/\left(\lvert x\rvert\sqrt{x^2-1}\right)$

Hyperbolic

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$

$f(x)$	$f'(x)$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

Inverse Hyperbolic

$f(x)$	$f'(x)$
$\sinh^{-1} x$	$1/\sqrt{x^2+1}$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$
$\tanh^{-1} x$	$1/(1-x^2),\ \lvert x\rvert < 1$

Special Techniques

Technique	Formula
Implicit: $F(x,y)=0$	$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$
Log diff: $y = f(x)^{g(x)}$	$\dfrac{dy}{dx} = y\left[g' \ln f + g\dfrac{f'}{f}\right]$
Parametric 1st order	$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$
Parametric 2nd order	$\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt}$

End of Chapter 4. This chapter has derived every rule from first principles and demonstrated each with engineering-context worked examples. Solutions to all exercises are available in the Solutions Manual.

CHAPTER 5

Higher-Order Derivatives

nth Derivatives • Concavity • Leibniz Rule • Beam Deflection • Vibrations

Higher-order derivatives arise naturally whenever we differentiate a function more than once. While the first derivative captures the instantaneous rate of change of a function, second and higher derivatives reveal deeper geometric, physical, and structural information that is indispensable in engineering analysis, mathematical physics, and applied science. The curvature of a beam under load, the acceleration of a vibrating machine, the shape of a profit curve, and the oscillatory decay of a damped spring are all governed by relationships that involve second, third, and fourth derivatives.

This chapter develops the theory of higher-order derivatives in a rigorous and systematic way. We begin with formal definitions and derive closed-form formulas for the nth derivative of all standard elementary functions — exponentials, trigonometric functions, logarithms, and power functions. We then explore the profound physical interpretations of the derivative hierarchy: velocity, acceleration, jerk, snap, and beyond. Section 5.3 applies the second derivative to the geometry of curves, establishing the theory of concavity and inflection points alongside the second derivative test for classifying critical points. Section 5.4 derives the powerful Leibniz product rule for the nth derivative — one of the most elegant results in differential calculus — and demonstrates its application through numerous worked examples. Finally, Section 5.5 applies the full theoretical machinery to two central areas of engineering: the deflection of beams governed by a fourth-order differential equation, and the analysis of mechanical vibrations described by second-order equations of motion. Throughout, derivations are carried out from first principles, and every result is verified and illustrated through original worked examples.

5.1 Second and Higher Derivatives

5.1.1 Definitions and Notation

Let f be a function that is differentiable on an open interval I. The first derivative $f'(x)$ is itself a function on I (or a subset thereof). If $f'(x)$ is also differentiable, we may differentiate it again to obtain the second derivative:

$$f''(x) = \frac{d}{dx}[f'(x)]$$

More generally, the nth derivative of f is defined recursively as:

$$f^{(n)}(x) = \frac{d}{dx}\left[f^{(n-1)}(x)\right], \qquad n = 1,2,3,\ldots$$

with the convention $f^{(0)}(x) = f(x)$. Three major notational systems are used throughout mathematics and engineering:

82. **Lagrange (prime) notation:** $f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x)$

83. **Leibniz notation:** $\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots,\ \frac{d^ny}{dx^n}$

84. **Newton dot notation (time derivatives):** $\dot{y},\ \ddot{y}$ (rarely used beyond the second)

In Leibniz notation, the second derivative is formally written as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

This notation is suggestive of algebraic manipulation but should be understood as an operator acting on f — not a ratio of infinitesimals. The Leibniz notation is preferred in engineering because it makes the variable of differentiation explicit, which is critical in multi-variable settings and differential equations.

For a function $y = f(x)$, the relationship between notations is:

$$f^{(2)}(x) = f''(x) = \frac{d^2y}{dx^2} \qquad \text{(second derivative)}$$

$$f^{(3)}(x) = f'''(x) = \frac{d^3y}{dx^3} \qquad \text{(third derivative)}$$

$$f^{(n)}(x) = \frac{d^ny}{dx^n} \qquad (n\text{th derivative})$$

5.1.2 Derivatives of Power Functions

Let $f(x) = x^m$ where m is a positive integer. Successive differentiation yields:

$$f'(x) = m\,x^{m-1}$$

$$f''(x) = m(m-1)\,x^{m-2}$$

$$f'''(x) = m(m-1)(m-2)\,x^{m-3}$$

$$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)\,x^{m-k} \qquad \text{for } k \le m$$

$$f^{(m)}(x) = m! \qquad \text{(constant)}$$

$$f^{(k)}(x) = 0 \qquad \text{for all } k > m$$

Using the falling factorial notation $m^{\underline{k}} = m!/(m-k)!$, the general formula becomes:

$$f^{(k)}(x) = \frac{m!}{(m-k)!}\, x^{m-k} \qquad (k \le m)$$

5.1.3 Derivation: nth Derivative of e^{ax}

Let $f(x) = e^{ax}$. We claim that $f^{(n)}(x) = a^n e^{ax}$ for all $n \ge 1$.

Proof by Mathematical Induction:

Base case ($n = 1$): $f'(x) = ae^{ax} = a^1 e^{ax}$. ✓

Inductive step: Assume $f^{(k)}(x) = a^k e^{ax}$ for some $k \ge 1$. Then:

$$f^{(k+1)}(x) = \frac{d}{dx}[a^k e^{ax}] = a^k \cdot a\, e^{ax} = a^{k+1} e^{ax} \quad \checkmark$$

Therefore, by induction, the formula holds for all $n \ge 1$.

$$\boxed{(e^{ax})^{(n)} = a^n e^{ax}}$$

5.1.4 Derivation: nth Derivative of $\sin(ax)$ and $\cos(ax)$

Let $f(x) = \sin(ax)$. The first four derivatives follow a repeating cycle of period 4:

$$f'(x) = a\cos(ax)$$

$$f''(x) = -a^2\sin(ax)$$

$$f'''(x) = -a^3\cos(ax)$$

$$f^{(4)}(x) = a^4\sin(ax) \qquad \leftarrow \text{identical to } f(x) \text{ scaled by } a^4$$

We claim the general formula is:

$$\boxed{(\sin ax)^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)}$$

Proof by Mathematical Induction:

Base case ($n = 1$): $a^1\sin(ax + \pi/2) = a\cos(ax)$. ✓ (since $\sin(\theta + \pi/2) = \cos\theta$)

Inductive step: Assume $f^{(k)}(x) = a^k\sin(ax + k\pi/2)$. Differentiating:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[a^k \sin\left(ax + \frac{k\pi}{2}\right)\right] = a^k \cdot a\cos\left(ax + \frac{k\pi}{2}\right)$$

$$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right) = a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right) \quad \checkmark$$

Similarly, for $f(x) = \cos(ax)$:

$$\boxed{(\cos ax)^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)}$$

Verification $(n = 1)$: $a\cos(ax + \pi/2) = a(-\sin ax) = -a\sin(ax)$. Since $\cos(\theta + \pi/2) = -\sin\theta$, this gives $(\cos ax)' = -a\sin(ax)$. ✓

5.1.5 Derivation: nth Derivative of $\ln x$

Let $f(x) = \ln x$, $x > 0$. Computing successive derivatives:

$$f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -6x^{-4}$$

The general pattern is:

$$\boxed{(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}} \qquad (n \geq 1)$$

Proof — Inductive step: Assume the formula holds for k. Then:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[(-1)^{k-1}(k-1)!\, x^{-k}\right] = (-1)^{k-1}(k-1)! \cdot (-k)\, x^{-k-1}$$

$$= (-1)^k\, k!\, x^{-(k+1)} \quad \checkmark$$

5.1.6 Reference Table: nth Derivatives of Common Functions

$f(x)$	$f^{(n)}(x)$	Conditions
x^m	$\frac{m!}{(m-n)!}\, x^{m-n}$	$n \leq m$; equals $m!$ when $n = m$; 0 when $n > m$
e^x	e^x	All n
e^{ax}	$a^n e^{ax}$	All n, a constant
$\sin(ax)$	$a^n \sin(ax + n\pi/2)$	All n
$\cos(ax)$	$a^n \cos(ax + n\pi/2)$	All n
$\ln x$	$(-1)^{n-1}(n-1)!/x^n$	$n \geq 1$, $x > 0$
$(1+x)^m$	$m(m-1)\cdots(m-n+1)(1+x)^{m-n}$	Any real m

$f(x)$	$f^{(n)}(x)$	Conditions
$1/(1-x)$	$n!/(1-x)^{n+1}$	$\lvert x\rvert < 1$

Worked Examples — Section 5.1

Example 5.1.1 — Polynomial Derivatives

Find the first six derivatives of $f(x) = 2x^6 - 5x^4 + 3x^2 - 7x + 4$.

Solution:

$$f'(x) = 12x^5 - 20x^3 + 6x - 7$$

$$f''(x) = 60x^4 - 60x^2 + 6$$

$$f'''(x) = 240x^3 - 120x$$

$$f^{(4)}(x) = 720x^2 - 120$$

$$f^{(5)}(x) = 1440x$$

$$f^{(6)}(x) = 1440 \qquad \text{(constant)}$$

$f^{(k)}(x) = 0$ for all $k \geq 7$ (a degree-6 polynomial vanishes after 6 differentiations).

■

Example 5.1.2 — nth Derivative of $e^x \sin x$

Find the general formula for $f^{(n)}(x)$ when $f(x) = e^x \sin x$.

Solution — using the complex representation method:

Write $e^x \sin x = \text{Im}\left(e^x \cdot e^{ix}\right) = \text{Im}\left(e^{(1+i)x}\right)$.

The nth derivative of $e^{(1+i)x}$ is $(1+i)^n e^{(1+i)x}$.

Convert $(1+i)$ to polar form: $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\theta = \arctan(1/1) = \pi/4$.

So $(1+i)^n = \left(\sqrt{2}\right)^n e^{in\pi/4} = 2^{n/2}\left(\cos(n\pi/4) + i\sin(n\pi/4)\right)$.

Therefore:

$$(e^x \sin x)^{(n)} = \text{Im}\left[2^{n/2}\, e^{(1+i)x}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right]$$

$$= 2^{n/2}\, e^x \text{Im}\left[e^{ix}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right]$$

$$= \boxed{2^{n/2}\, e^x \sin\left(x + \frac{n\pi}{4}\right)}$$

Verification ($n = 1$): $\sqrt{2}\,e^x \sin(x + \pi/4) = \sqrt{2}\,e^x\left[\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right] = e^x(\sin x + \cos x)$.

Direct check: $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$. ✓

■

Example 5.1.3 — $f^{(4)}(x)$ for $x^3 \ln x$

Find $f^{(4)}(x)$ for $f(x) = x^3 \ln x$.

Solution — Successive differentiation:

$$f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \ln x + x^2$$

$$f''(x) = 6x \ln x + 3x^2 \cdot \frac{1}{x} + 2x = 6x \ln x + 3x + 2x = 6x \ln x + 5x$$

$$f'''(x) = 6 \ln x + 6x \cdot \frac{1}{x} + 5 = 6 \ln x + 11$$

$$f^{(4)}(x) = \frac{6}{x}$$

Note: All higher derivatives follow: $f^{(5)}(x) = -6/x^2$, $f^{(6)}(x) = 12/x^3$, …

■

Exercises 5.1

5.1.1 Find $f^{(5)}(x)$ for $f(x) = 3x^7 - 2x^5 + x^3 - 4x + 9$.

5.1.2 Find $f^{(n)}(x)$ for each: (a) $f(x) = e^{-3x}$ (b) $f(x) = \sin(4x)$ (c) $f(x) = \cos(x/3)$ (d) $f(x) = \ln(2x)$

5.1.3 Find $f^{(4)}(x)$ for $f(x) = x^2 e^x$ using successive differentiation.

5.1.4 Derive the general formula for the nth derivative of $f(x) = 1/(1 - x)$, $|x| < 1$. [*Hint:* Write as $(1 - x)^{-1}$ and differentiate repeatedly. Verify against the geometric series.]

5.1.5 Find the nth derivative of $f(x) = xe^x$. [Answer: $f^{(n)}(x) = (x + n)e^x$]

5.1.6 Prove by induction that $(\cos ax)^{(n)} = a^n \cos(ax + n\pi/2)$.

5.2 Physical Meaning: Acceleration, Jerk, and Beyond

5.2.1 The Kinematic Hierarchy of Derivatives

In classical mechanics, the motion of a particle along a straight line is completely described by its position function $s(t)$, giving displacement from a reference point as a function of time t. Each successive derivative of $s(t)$ carries a distinct, physically meaningful interpretation, forming a natural hierarchy:

Derivative	Symbol	Name	SI Unit	Physical Meaning
$s(t)$	s	Position / Displacement	m	Location relative to reference
$s'(t) = ds/dt$	$v(t)$	Velocity	m/s	Rate of change of position
$s''(t) = d^2s/dt^2$	$a(t)$	Acceleration	m/s^2	Rate of change of velocity
$s'''(t) = d^3s/dt^3$	$j(t)$	Jerk (Jolt)	m/s^3	Rate of change of acceleration
$s^{(4)}(t) = d^4s/dt^4$	snap(t)	Snap (Jounce)	m/s^4	Rate of change of jerk
$s^{(5)}(t)$	crackle(t)	Crackle	m/s^5	Rate of change of snap
$s^{(6)}(t)$	pop(t)	Pop	m/s^6	Rate of change of crackle

Velocity $v(t) = s'(t)$ measures how rapidly the particle's position is changing and can be positive, negative, or zero. Acceleration $a(t) = s''(t) = v'(t)$ measures how the velocity is changing — it is directly related to force by Newton's second law: $F = ma$. Jerk $j(t) = s'''(t)$ is the third derivative and quantifies abruptness of motion. Jerk is a critical engineering parameter in the design of elevators, railway systems, robotic arms, and vehicles, as humans are sensitive to sudden changes in acceleration. High jerk creates mechanical stress and passenger discomfort.

The fourth derivative, snap or jounce, appears in precision trajectory planning in robotics and aerospace engineering. The fifth and sixth derivatives (crackle and pop) arise primarily in satellite attitude control and semiconductor manufacturing robots, where trajectories must satisfy higher-order continuity constraints to avoid inducing vibrations in ultra-sensitive systems.

5.2.2 Sign Interpretation and Speed

An important distinction exists between velocity and speed: speed is the magnitude of velocity ($|v(t)|$) and is always non-negative. A particle speeds up when its velocity and acceleration have the same sign, and slows down when they have opposite signs:

Condition	Physical Interpretation
$v(t) > 0$	Moving in the positive direction

Condition	Physical Interpretation
$v(t) < 0$	Moving in the negative direction
$v(t) = 0$	Momentarily at rest
$a(t) > 0$	Velocity is increasing
$a(t) < 0$	Velocity is decreasing
$v(t)$ and $a(t)$ same sign	Speeding up (speed increasing)
$v(t)$ and $a(t)$ opposite sign	Slowing down (speed decreasing)

5.2.3 Simple Harmonic Motion and Its Derivatives

For a particle executing simple harmonic motion (SHM):

$$s(t) = A\sin(\omega t + \varphi)$$

where A is amplitude (m), ω is angular frequency (rad/s), and φ is the initial phase angle (rad). The full derivative hierarchy is:

$$v(t) = s'(t) = A\omega\cos(\omega t + \varphi)$$

$$a(t) = s''(t) = -A\omega^2\sin(\omega t + \varphi) = -\omega^2 s(t)$$

$$j(t) = s'''(t) = -A\omega^3\cos(\omega t + \varphi) = -\omega^2 v(t)$$

The key relation $a(t) = -\omega^2 s(t)$ is the defining differential equation of SHM, showing that the restoring acceleration is proportional to — and oppositely directed to — displacement. This will be the cornerstone of Section 5.5.2.

5.2.4 Free Fall Under Gravity

For an object launched vertically with initial velocity v_0 from height s_0, ignoring air resistance:

$$s(t) = s_0 + v_0 t - 1/2\, g t^2$$

where $g = 9.81$ m/s^2 is the acceleration due to gravity.

$$v(t) = v_0 - gt \qquad a(t) = -g \text{ (constant)} \qquad j(t) = 0$$

The zero jerk in free fall reflects the fact that gravity is a constant force. In contrast, rocket propulsion produces nonzero jerk because thrust varies with time.

Worked Examples — Section 5.2

Example 5.2.1 — Particle Motion Analysis

A particle moves along the x-axis with position:

$$s(t) = t^3 - 6t^2 + 9t + 2 \qquad \text{(metres, } t \text{ in seconds, } t \geq 0)$$

85. Find velocity, acceleration, and jerk functions. (b) Find when the particle is at rest. (c) Find when the particle is speeding up and slowing down. (d) Find the total distance travelled in the first 4 seconds.

Solution:

$$v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$$

$$a(t) = s''(t) = 6t - 12 = 6(t-2)$$

$$j(t) = s'''(t) = 6 \text{ m/s}^3 \quad \text{(constant)}$$

86. At rest: $v(t) = 0 \Longrightarrow 3(t-1)(t-3) = 0 \Longrightarrow t = 1$ s and $t = 3$ s.

87. Sign table for $v(t)$ and $a(t)$:

88. $(0,1)$: $v(0.5) = 3.75 > 0$; $a(0.5) = -9 < 0 \rightarrow$ **Slowing down**

89. $(1,2)$: $v(1.5) = -2.25 < 0$; $a(1.5) = -3 < 0 \rightarrow$ **Speeding up**

90. $(2,3)$: $v(2.5) = -2.25 < 0$; $a(2.5) = 3 > 0 \rightarrow$ **Slowing down**

91. $(3,4)$: $v(3.5) = 3.75 > 0$; $a(3.5) = 9 > 0 \rightarrow$ **Speeding up**

92. Positions at key times: $s(0) = 2$ m, $s(1) = 6$ m, $s(3) = 2$ m, $s(4) = 6$ m.

Total distance $= |s(1) - s(0)| + |s(3) - s(1)| + |s(4) - s(3)| = 4 + 4 + 4 = 12$ m

■

Example 5.2.2 — Engineering Vehicle Test

An engineering test vehicle's position is $s(t) = 60t - 3t^2 + 1/9\, t^3$ (m, $0 \le t \le 15$ s).

93. Find velocity, acceleration, and jerk. (b) Find when acceleration equals zero. (c) Find velocity at the instant of zero acceleration. (d) Calculate the jerk at $t = 6$ s and comment on its engineering significance.

Solution:

$$v(t) = 60 - 6t + 1/3\, t^2 \qquad a(t) = -6 + 2/3\, t \qquad j(t) = 2/3 \text{ m/s}^3 \text{ (constant)}$$

94. $a(t) = 0$: $-6 + 2/3\, t = 0 \Longrightarrow t = 9$ s.

95. $v(9) = 60 - 54 + 1/3\,(81) = 60 - 54 + 27 = 33$ m/s.

96. $j(6) = 2/3$ m/s^3 (same everywhere — constant jerk profile throughout the test).

Engineering significance: A constant jerk profile is often deliberately designed into vehicle acceleration systems (called a "jerk-limited" profile) because it produces smooth, predictable acceleration changes that minimise passenger discomfort and mechanical wear on drivetrain components.

■

Example 5.2.3 — Damped Vibration Derivatives

The displacement of a shock absorber is $s(t) = e^{-2t}\sin(5t)$ metres.

Find the velocity, acceleration, and jerk. Evaluate jerk at $t = 0$.

Solution:

$$v(t) = s'(t) = -2e^{-2t}\sin(5t) + 5e^{-2t}\cos(5t) = e^{-2t}(-2\sin 5t + 5\cos 5t)$$

$$a(t) = v'(t) = \frac{d}{dt}\left[e^{-2t}(-2\sin 5t + 5\cos 5t)\right]$$

$$= -2e^{-2t}(-2\sin 5t + 5\cos 5t) + e^{-2t}(-10\sin 5t - 10\cos 5t)$$

$$= e^{-2t}[(4\sin 5t - 10\cos 5t) + (-10\sin 5t - 10\cos 5t)]$$

$$= e^{-2t}(-6\sin 5t - 20\cos 5t) = -2e^{-2t}(3\sin 5t + 10\cos 5t)$$

For the jerk, differentiate $a(t)$ using the product rule:

$$j(t) = a'(t) = \frac{d}{dt}\left[-2e^{-2t}(3\sin 5t + 10\cos 5t)\right]$$

$$= -2[-2e^{-2t}(3\sin 5t + 10\cos 5t) + e^{-2t}(15\cos 5t - 50\sin 5t)]$$

$$= -2e^{-2t}[-6\sin 5t - 20\cos 5t + 15\cos 5t - 50\sin 5t]$$

$$= -2e^{-2t}(-56\sin 5t - 5\cos 5t) = 2e^{-2t}(56\sin 5t + 5\cos 5t)$$

At $t = 0$:

$$j(0) = 2e^{0}(56 \cdot 0 + 5 \cdot 1) = 2 \times 5 = 10 \text{ m/s}^3$$

■

Exercises 5.2

5.2.1 A rocket's altitude is $h(t) = 80t^2 - 8/3\,t^3$ (m, $0 \le t \le 20$ s). Find: (a) velocity at $t = 4$ s; (b) acceleration at $t = 6$ s; (c) when velocity is maximum; (d) the jerk function and its value.

5.2.2 A piston follows $x(t) = 0.08\cos(40\pi t)$ metres. Find: (a) maximum velocity; (b) maximum acceleration; (c) jerk at $t = 1/80$ s.

5.2.3 Show that for $s(t) = A\sin(\omega t + \varphi)$, the acceleration satisfies $a = -\omega^2 s$. Explain the physical significance of this proportionality.

5.2.4 A particle's acceleration is $a(t) = 6t - 18$ m/s^2. If $v(0) = 4$ m/s and $s(0) = 0$, find $s(t)$ and the jerk at any time t.

5.2.5 A bungee cord gives a jumper the position $s(t) = 50 - 25e^{-0.5t}\cos(2t)$ m. Find the velocity and acceleration at $t = 0$ and interpret physically.

5.3 Concavity and Inflection Points

5.3.1 Definition of Concavity

The second derivative provides precise quantitative information about the shape — specifically the curvature — of a function's graph. This geometric property is called concavity.

Definition: A function f is said to be:

97. **Concave up** on an interval I if $f''(x) > 0$ for all $x \in I$.

98. **Concave down** on an interval I if $f''(x) < 0$ for all $x \in I$.

Geometric interpretation:

Concave up: The graph lies above all its tangent lines. The slope $f'(x)$ is increasing. Visually, the curve looks like the inside of a bowl (∪).

Concave down: The graph lies below all its tangent lines. The slope $f'(x)$ is decreasing. Visually, the curve looks like an upside-down bowl (∩).

5.3.2 Proof that $f'' > 0$ Implies Concave Up

Theorem: If $f''(x) > 0$ on I, then the graph of f lies above every tangent line on I.

Proof: Let $a \in I$. The tangent line at $\big(a, f(a)\big)$ is $L(x) = f(a) + f'(a)(x - a)$.

We want to show $f(x) > L(x)$ for $x \neq a$. By Taylor's theorem with Lagrange remainder:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c)}{2}(x - a)^2$$

for some c strictly between a and x. Therefore:

$$f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2$$

Since $f''(c) > 0$ (by assumption) and $(x - a)^2 > 0$ for $x \neq a$, we conclude $f(x) - L(x) > 0$, i.e. $f(x) > L(x)$ for all $x \neq a$. ▫

5.3.3 Inflection Points — Definition and Method

Definition: A point $\big(c, f(c)\big)$ is an *inflection point* of f if (i) f is continuous at c, and (ii) the concavity of f changes sign at c — that is, f'' changes from positive to negative or vice versa at $x = c$.

Necessary condition: If $\big(c, f(c)\big)$ is an inflection point and $f''(c)$ exists, then $f''(c) = 0$.

IMPORTANT CAUTION: $f''(c) = 0$ is necessary but **NOT** sufficient. We must verify the sign change. For example: $f(x) = x^4$ has $f''(0) = 0$ but does **NOT** have an inflection point at $x = 0$ because $f''(x) = 12x^2 \geq 0$ on both sides of 0 — no sign change occurs.

Procedure for finding inflection points:

Step 1: Compute $f''(x)$.

Step 2: Solve $f''(x) = 0$ and identify points where f'' is undefined.

Step 3: Divide the real line into intervals at these candidate points.

Step 4: Test the sign of f'' in each interval.

Step 5: An inflection point exists at $x = c$ if and only if f'' changes sign at c.

5.3.4 The Second Derivative Test for Local Extrema

Theorem (Second Derivative Test): Suppose $f'(c) = 0$ and $f''(c)$ exists.

99. If $f''(c) > 0$, then f has a **local minimum** at c.

100. If $f''(c) < 0$, then f has a **local maximum** at c.

101. If $f''(c) = 0$, the test is **inconclusive** — use the first derivative test.

Proof of the minimum case: Suppose $f'(c) = 0$ and $f''(c) > 0$. Since $f''(c) > 0$, by definition of the derivative:

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h} > 0$$

This means for sufficiently small $|h| > 0$: $f'(c+h)/h > 0$. For $h > 0$ this gives $f'(c+h) > 0$ (increasing to the right of c), and for $h < 0$ this gives $f'(c+h) < 0$ (decreasing to the left of c). By the First Derivative Test, f has a local minimum at c. ▫

5.3.5 Higher-Order Derivative Test (Inconclusive Cases)

When $f'(c) = 0$ and $f''(c) = 0$ simultaneously, we examine higher derivatives:

Theorem: Suppose $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. Then:

102. If n is **even** and $f^{(n)}(c) > 0$: **local minimum** at c.

103. If n is **even** and $f^{(n)}(c) < 0$: **local maximum** at c.

104. If n is **odd**: **neither** (a horizontal inflection point).

Example: $f(x) = x^5$. We have $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0$, but $f^{(5)}(0) = 120 \neq 0$ and $n = 5$ is odd. Therefore $x = 0$ is a horizontal inflection point — consistent with the graph of x^5.

Worked Examples — Section 5.3

Example 5.3.1 — Full Curve Analysis

Analyse $f(x) = x^4 - 8x^3 + 18x^2 - 10$ for concavity, inflection points, and extrema.

Solution:

$$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$$

$$f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-1)(x-3)$$

Critical points ($f'(x) = 0$): $x = 0$ and $x = 3$.

$f''(0) = 12(-1)(-3) = 36 > 0 \rightarrow$ **local minimum** at $x = 0$. $f(0) = -10$.

$f''(3) = 0 \rightarrow$ inconclusive; use higher-order test.

$f'''(x) = 24x - 48$; $f'''(3) = 24 \neq 0$, so $n = 3$ (odd) $\rightarrow x = 3$ is a **horizontal inflection point**.

Inflection points ($f''(x) = 0$): $x = 1$ and $x = 3$.

Sign of $f'' = 12(x-1)(x-3)$:

105. $x < 1$: $(+)(-)(-) = (+) > 0 \rightarrow$ concave up

106. $1 < x < 3$: $(+)(+)(-) = (-) < 0 \rightarrow$ concave down

107. $x > 3$: $(+)(+)(+) = (+) > 0 \rightarrow$ concave up

f'' changes sign at both $x = 1$ and $x = 3 \rightarrow$ both are inflection points.

$f(1) = 1 - 8 + 18 - 10 = 1$; point $(1, 1)$ is an inflection point.

$f(3) = 81 - 216 + 162 - 10 = 17$; point $(3, 17)$ is the horizontal inflection point.

Summary: Local minimum at $(0, -10)$. Inflection points at $(1, 1)$ and $(3, 17)$.

■

Example 5.3.2 — Inflection Points of a Transcendental Function

Find the concavity and inflection points of $f(x) = xe^{-x}$.

Solution:

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$$

$$f''(x) = -e^{-x}(1 - x) + e^{-x}(-1) = e^{-x}(-1 + x - 1) = e^{-x}(x - 2)$$

Since $e^{-x} > 0$ always, the sign of f'' is determined entirely by $(x - 2)$:

108. $x < 2$: $f''(x) < 0 \rightarrow$ concave down

109. $x > 2$: $f''(x) > 0 \rightarrow$ concave up

110. $x = 2$: $f''(2) = 0$ and f'' changes sign $\rightarrow$ inflection point

$$f(2) = 2e^{-2} \approx 0.2707$$

Inflection point at $(2,\ 2e^{-2}) \approx (2,\ 0.271)$**.**

■

Example 5.3.3 — Optimisation with Second Derivative Test

A manufacturing company's weekly profit (thousands of dollars) is:

$$P(x) = -2x^3 + 15x^2 - 24x + 5 \qquad (x = \text{units produced, hundreds})$$

Find the production level that maximises profit and verify using the second derivative test.

Solution:

$$P'(x) = -6x^2 + 30x - 24 = -6(x^2 - 5x + 4) = -6(x - 1)(x - 4)$$

$P'(x) = 0 \Longrightarrow x = 1$ and $x = 4$.

$$P''(x) = -12x + 30$$

$P''(1) = 18 > 0 \rightarrow$ **local minimum** at $x = 1$. $P(1) = -2 + 15 - 24 + 5 = -6$.

$P''(4) = -18 < 0 \rightarrow$ **local maximum** at $x = 4$. $P(4) = -128 + 240 - 96 + 5 = 21$.

Inflection point ($P''(x) = 0$): $x = 2.5$.

$$P(2.5) = -2(15.625) + 15(6.25) - 24(2.5) + 5 = -31.25 + 93.75 - 60 + 5 = 7.5$$

Maximum weekly profit of \$21,000 is achieved at $x = 4$ **(400 units per week).** The inflection point at $(2.5,\ 7.5)$ marks where the rate of profit growth switches from increasing to decreasing — the "diminishing returns" threshold.

■

Exercises 5.3

5.3.1 Find intervals of concavity and inflection points for: (a) $f(x) = x^3 - 3x^2 + 4$; (b) $f(x) = \sin x$ on $[0, 2\pi]$; (c) $f(x) = x^2 e^{-x}$.

5.3.2 Apply the second derivative test to classify all critical points of $f(x) = 3x^4 - 16x^3 + 24x^2 - 5$.

5.3.3 Show that $f(x) = x^4$ has a local minimum at $x = 0$ even though $f''(0) = 0$. Use the higher-order derivative test and verify graphically.

5.3.4 For $f(x) = x^5 - 5x$, find all local extrema and inflection points.

5.3.5 A beam's bending moment is $M(x) = 15x - x^2 - 20$ (kN·m, x in m, $0 \le x \le 10$). Find the location of maximum bending moment and classify it using the second derivative test.

5.3.6 Prove that a polynomial of odd degree with positive leading coefficient must have at least one inflection point.

5.4 The nth Derivative and the Leibniz Rule

5.4.1 The Challenge of nth Derivatives of Products

For the nth derivative of a product of two functions, repeated application of the ordinary product rule quickly becomes cumbersome. For example, finding $(x^4 \sin x)^{(8)}$ by successive differentiation would require eight applications of the product rule. The Leibniz Rule provides an elegant closed-form formula that solves this problem in one step, using the same structure as the Binomial Theorem.

5.4.2 Statement of the Leibniz Rule

Theorem (General Leibniz Rule): If f and g are n times differentiable functions on an interval I, then their product fg is also n times differentiable, and:

$$\boxed{(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ are the binomial coefficients. Written in expanded form:

$$(fg)^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2!}f^{(n-2)}g'' + \cdots + nf\,g^{(n-1)} + fg^{(n)}$$

5.4.3 Derivation by Mathematical Induction

Proof:

Base case ($n = 1$): The formula gives $\binom{1}{0}f^{(0)}g^{(1)} + \binom{1}{1}f^{(1)}g^{(0)} = fg' + f'g$, which is exactly the standard product rule. ✓

Inductive step: Assume the formula holds for $n = k$:

$$(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$$

Differentiating both sides with respect to x:

$$(fg)^{(k+1)} = \sum_{j=0}^{k} \binom{k}{j} \left[f^{(j+1)} g^{(k-j)} + f^{(j)} g^{(k-j+1)} \right]$$

Splitting into two sums and re-indexing the first (let $m = j + 1$):

$$= \sum_{m=1}^{k+1} \binom{k}{m-1} f^{(m)} g^{(k+1-m)} + \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k+1-j)}$$

Collecting terms for each m from 0 to $k + 1$ and applying Pascal's identity $\binom{k}{m-1} + \binom{k}{m} = \binom{k+1}{m}$, we obtain:

$$(fg)^{(k+1)} = \sum_{m=0}^{k+1} \binom{k+1}{m} f^{(m)} g^{(k+1-m)}$$

This is precisely the Leibniz formula for $n = k + 1$. ▫

The structural identity with the Binomial Theorem is not coincidental — both arise from the same combinatorial counting argument, with differentiation replacing multiplication.

5.4.4 Strategy for Applying the Leibniz Rule

Assign f to the function whose derivatives eventually become zero or simplify significantly (typically a polynomial), and g to the function whose derivatives are easy to compute in general (typically an exponential or trigonometric function). This minimises the number of nonzero terms in the sum.

Worked Examples — Section 5.4

Example 5.4.1 — Leibniz Rule: Polynomial × Exponential

Find the nth derivative of $h(x) = x^3 e^x$.

Solution: Let $f(x) = x^3$ (polynomial) and $g(x) = e^x$.

Derivatives of f: $f' = 3x^2$, $f'' = 6x$, $f''' = 6$, $f^{(k)} = 0$ for $k \geq 4$.

Derivatives of g: $g^{(k)}(x) = e^x$ for all k.

By the Leibniz rule (only $k = 0,1,2,3$ contribute since $f^{(k)} = 0$ for $k \geq 4$):

$$h^{(n)}(x) = \binom{n}{0} x^3 e^x + \binom{n}{1} 3x^2 e^x + \binom{n}{2} 6x\, e^x + \binom{n}{3} 6\, e^x$$

$$= e^x[x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2)] \qquad (n \geq 3)$$

Verification $(n = 2)$: $h''(x) = e^x(x^3 + 6x^2 + 6x)$.

Direct check: $h'(x) = e^x(x^3 + 3x^2)$, $h''(x) = e^x(x^3 + 3x^2) + e^x(3x^2 + 6x) = e^x(x^3 + 6x^2 + 6x)$. ✓

■

Example 5.4.2 — Leibniz Rule: Polynomial × Sine

Find $h^{(5)}(x)$ for $h(x) = x^2 \sin x$.

Solution: Let $f(x) = x^2$ and $g(x) = \sin x$.

Derivatives of f: $f = x^2$, $f' = 2x$, $f'' = 2$, $f^{(k)} = 0$ for $k \geq 3$.

Derivatives of g: $g^{(k)}(x) = \sin(x + k\pi/2)$.

$$g^{(5)}(x) = \sin(x + 5\pi/2) = \sin(x + 2\pi + \pi/2) = \cos x$$

$$g^{(4)}(x) = \sin(x + 4\pi/2) = \sin(x + 2\pi) = \sin x$$

$$g^{(3)}(x) = \sin(x + 3\pi/2) = -\cos x$$

Applying Leibniz with $n = 5$:

$$h^{(5)}(x) = \binom{5}{0} x^2 \cos x + \binom{5}{1} 2x \sin x + \binom{5}{2} 2(-\cos x)$$

$$= x^2 \cos x + 10x \sin x - 20 \cos x$$

$$= \boxed{(x^2 - 20)\cos x + 10x \sin x}$$

■

Example 5.4.3 — General Formula: $e^x \cos x$

Derive the general nth derivative formula for $h(x) = e^x \cos x$.

Solution — using polar form of complex numbers:

Write $e^x \cos x = \mathrm{Re}\left(e^x \cdot e^{ix}\right) = \mathrm{Re}\left(e^{(1+i)x}\right)$.

The nth derivative of $e^{(1+i)x}$ is $(1 + i)^n e^{(1+i)x}$.

In polar form: $(1 + i) = \sqrt{2}\, e^{i\pi/4}$, so $(1 + i)^n = 2^{n/2} e^{in\pi/4}$.

Therefore:

$$h^{(n)}(x) = \mathrm{Re}\left[2^{n/2} e^{in\pi/4} \cdot e^{(1+i)x}\right] = \mathrm{Re}\left[2^{n/2} e^x\, e^{i(x+n\pi/4)}\right]$$

$$= \boxed{2^{n/2}\, e^{x} \cos\left(x + \frac{n\pi}{4}\right)}$$

Verification ($n = 1$): $\sqrt{2}\, e^{x}\cos(x + \pi/4) = \sqrt{2}\, e^{x}\left[\frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}}\right] = e^{x}(\cos x - \sin x)$.

Direct: $h'(x) = e^{x}\cos x - e^{x}\sin x = e^{x}(\cos x - \sin x)$. ✓

Verification ($n = 2$): $2e^{x}\cos(x + \pi/2) = 2e^{x}(-\sin x) = -2e^{x}\sin x$.

Direct: $h''(x) = (e^{x}\cos x - e^{x}\sin x)' = e^{x}\cos x - e^{x}\sin x - e^{x}\sin x - e^{x}\cos x = -2e^{x}\sin x$. ✓

■

Example 5.4.4 — Leibniz Applied to $x^4 e^{2x}$

Find $h^{(6)}(x)$ for $h(x) = x^4 e^{2x}$.

Solution: $f = x^4$, $g = e^{2x}$, $g^{(k)} = 2^k e^{2x}$.

Derivatives of f: $f' = 4x^3$, $f'' = 12x^2$, $f''' = 24x$, $f^{(4)} = 24$, $f^{(k)} = 0$ for $k \geq 5$.

With $n = 6$ (only $k = 0,1,2,3,4$ contribute):

$$h^{(6)} = \binom{6}{0} x^4 \cdot 64e^{2x} + \binom{6}{1} 4x^3 \cdot 32e^{2x} + \binom{6}{2} 12x^2 \cdot 16e^{2x} + \binom{6}{3} 24x \cdot 8e^{2x} + \binom{6}{4} 24 \cdot 4e^{2x}$$

$$= e^{2x}[64x^4 + 768x^3 + 2880x^2 + 3840x + 1440]$$

$$= 32e^{2x}[2x^4 + 24x^3 + 90x^2 + 120x + 45]$$

■

Exercises 5.4

5.4.1 Use the Leibniz rule to find $h^{(n)}(x)$ for: (a) $h(x) = x^2 e^{x}$; (b) $h(x) = x^3 e^{-x}$; (c) $h(x) = x\sin x$.

5.4.2 Find the 6th derivative of $h(x) = x^3\cos(2x)$ using the Leibniz rule.

5.4.3 Derive the general formula $(e^{x}\sin x)^{(n)} = 2^{n/2} e^{x}\sin(x + n\pi/4)$.

5.4.4 Verify the Leibniz rule for $n = 3$ applied to $h(x) = x^2 \ln x$ by: (a) using the Leibniz formula directly; (b) direct successive differentiation.

5.4.5 Find $h^{(8)}(0)$ for $h(x) = x^4\cos x$. [*Hint:* use Leibniz; evaluate at $x = 0$.]

5.4.6 *(Challenge)* Use the Leibniz rule to prove that if $p(x)$ is a degree-m polynomial and $n > m$, then $\big(p(x)\, g(x)\big)^{(n)} = \sum_{k=0}^{m} \binom{n}{k} p^{(k)}(x)\, g^{(n-k)}(x)$.

5.5 Engineering Applications

5.5.1 Beam Deflection — The Euler-Bernoulli Theory

Background and Fundamental Assumptions

The Euler-Bernoulli beam theory is the classical framework for analysing the transverse bending of slender structural members and represents one of the most important applications of fourth-order differential equations in civil and mechanical engineering. The theory was developed by Daniel Bernoulli and Leonhard Euler in the 18th century and remains the foundation of structural analysis to this day.

The theory rests on five fundamental assumptions:

111. The beam is initially straight, prismatic (uniform cross-section), and linearly elastic.

112. Deflections are small compared to the beam's length (small-deflection, or linear, theory).

113. The material is homogeneous and isotropic, obeying Hooke's Law.

114. Plane cross-sections remain plane and perpendicular to the neutral axis after bending (Bernoulli's hypothesis).

115. Transverse loads act in the plane of symmetry of the cross-section.

The Moment-Curvature Relationship

For a beam deflecting in the xy-plane, the exact curvature at any point is:

$$\kappa = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

Under the small-deflection assumption, $|dy/dx| \ll 1$, so $(dy/dx)^2 \approx 0$ and the curvature simplifies to:

$$\kappa \approx \frac{d^2y}{dx^2} = y''$$

The bending moment M is related to curvature by the fundamental moment-curvature law:

$$M(x) = EI\, y''(x)$$

where E is the elastic (Young's) modulus [Pa = N/m^2] and I is the second moment of area of the cross-section about its neutral axis [m^4]. The product EI is called the **flexural rigidity** of the beam.

Derivation of the Complete Governing Equation

From statics of an infinitesimal beam element, two equilibrium relationships hold:

$$\frac{dM}{dx} = V(x) \qquad \text{(shear force equals rate of change of moment)}$$

$$\frac{dV}{dx} = -w(x) \qquad \text{(distributed load equals rate of change of shear)}$$

Differentiating $M = EI\,y''$ with respect to x:

$$EI\,y''' = V(x)$$

Differentiating again:

$$EI\,y^{(4)} = -w(x)$$

The Euler-Bernoulli Beam Equation:

$$\boxed{EI\,\frac{d^4y}{dx^4} = w(x)}$$

(using the convention that downward distributed load w is positive and upward deflection y is positive; some textbooks reverse this sign convention).

This 4th-order ODE requires four boundary conditions. The complete hierarchy of derivatives and their physical meaning is:

Derivative	Quantity	Symbol	Physical Meaning
$y(x)$	Deflection	[m]	Transverse displacement of beam axis
$y'(x)$	Slope	[rad]	Angle of inclination of elastic curve
$EI \cdot y''(x)$	Bending Moment	$M(x)$ [N·m]	Internal moment — causes bending stress
$EI \cdot y'''(x)$	Shear Force	$V(x)$ [N]	Internal transverse force
$EI \cdot y^{(4)}(x)$	Distributed Load	$w(x)$ [N/m]	External load per unit length

Boundary Conditions for Common Support Types

Support Type	Boundary Conditions at $x = a$
Simply supported (pin/roller)	$y(a) = 0$ (no deflection), $y''(a) = 0$ (no moment)
Fixed / Clamped (cantilever wall)	$y(a) = 0$ (no deflection), $y'(a) = 0$ (no slope)
Free end	$y''(a) = 0$ (no moment), $y'''(a) = 0$ (no shear)
Free end with point load P	$y''(a) = 0$, $EI \cdot y'''(a) = \mp P$ (shear = applied load)

Worked Examples — Section 5.5.1

Example 5.5.1 — Simply Supported Beam with Uniform Load

Derive the deflection curve for a simply supported beam of length L carrying a uniformly distributed load of intensity w_0 (N/m). Find the maximum deflection.

Setup: $EI\,y^{(4)} = w_0$ (constant, downward load positive, upward y positive).

BCs: $y(0) = 0,\ y''(0) = 0,\ y(L) = 0,\ y''(L) = 0$.

Step 1 — Integrate four times:

$$EI\,y''' = w_0 x + C_1$$

$$EI\,y'' = \frac{w_0}{2}x^2 + C_1 x + C_2$$

$$EI\,y' = \frac{w_0}{6}x^3 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$EI\,y = \frac{w_0}{24}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

Step 2 — Apply boundary conditions:

BC1: $y(0) = 0 \Longrightarrow C_4 = 0$.

BC2: $y''(0) = 0 \Longrightarrow C_2 = 0$.

BC3: $y''(L) = 0 \Longrightarrow \frac{w_0}{2}L^2 + C_1 L = 0 \Longrightarrow C_1 = -\frac{w_0 L}{2}$.

BC4: $y(L) = 0$:

$$\frac{w_0}{24}L^4 - \frac{w_0 L}{2}\cdot\frac{L^3}{6} + C_3 L = 0 \Longrightarrow \frac{w_0 L^4}{24} - \frac{w_0 L^4}{12} + C_3 L = 0 \Longrightarrow C_3 = \frac{w_0 L^3}{24}$$

Step 3 — Deflection equation:

$$EI\,y = \frac{w_0}{24}x^4 - \frac{w_0 L}{12}x^3 + \frac{w_0 L^3}{24}x = \frac{w_0 x}{24}(x^3 - 2Lx^2 + L^3)$$

$$\boxed{y(x) = \frac{w_0 x(x^3 - 2Lx^2 + L^3)}{24EI}}$$

Step 4 — Maximum deflection at $x = L/2$ (by symmetry):

$$y\left(\frac{L}{2}\right) = \frac{w_0(L/2)}{24EI}\left[\frac{L^3}{8} - \frac{L^3}{2} + L^3\right] = \frac{w_0 L}{48EI}\cdot\frac{5L^3}{8}$$

$$\boxed{|y_{\max}| = \frac{5w_0 L^4}{384EI}} \qquad \leftarrow \text{standard result}$$

■

Example 5.5.2 — Cantilever Beam: Point Load at Free End

A cantilever of length L is clamped at $x = 0$ and free at $x = L$. A downward point load P is applied at the free end. Find the deflection curve.

Setup: No distributed load: $EI\, y^{(4)} = 0$.

BCs: $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$ (no moment at free end), $EI\, y'''(L) = P$ (shear = P).

Integrating $EI\, y^{(4)} = 0$ four times:

$$EI\, y''' = C_1, \quad EI\, y'' = C_1 x + C_2, \quad EI\, y' = \frac{C_1}{2}x^2 + C_2 x + C_3, \quad EI\, y = \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

Applying BCs:

$y(0) = 0 \Longrightarrow C_4 = 0; \quad y'(0) = 0 \Longrightarrow C_3 = 0.$

$EI\, y'''(L) = P \Longrightarrow C_1 = P; \quad y''(L) = 0 \Longrightarrow PL + C_2 = 0 \Longrightarrow C_2 = -PL.$

Deflection equation:

$$EI\, y = \frac{P}{6}x^3 - \frac{PL}{2}x^2 = \frac{Px^2}{6}(x - 3L)$$

$$\boxed{y(x) = \frac{Px^2(x - 3L)}{6EI}} \qquad \text{(downward deflection for } 0 \le x \le L\text{)}$$

Maximum deflection at $x = L$:

$$y(L) = \frac{PL^2(L - 3L)}{6EI} = \frac{-2PL^3}{6EI} \Longrightarrow \boxed{|y_{\max}| = \frac{PL^3}{3EI}} \qquad \leftarrow \text{standard cantilever result}$$

Slope at free end:

$$EI\, y'(L) = P \cdot \frac{L^2}{2} - PL \cdot L = -\frac{PL^2}{2} \Longrightarrow \theta_{\max} = \frac{PL^2}{2EI} \qquad \text{(rotation at free end)}$$

■

Example 5.5.3 — Fixed-Fixed Beam: Uniform Load

A beam is clamped at both ends ($x = 0$ and $x = L$) with uniform load w_0. Find $y(x)$.

BCs: $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y'(L) = 0$.

Integrating $EI\, y^{(4)} = w_0$:

$$EI\, y = \frac{w_0}{24}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

BC1: $y(0) = 0 \Longrightarrow C_4 = 0$; BC2: $y'(0) = 0 \Longrightarrow C_3 = 0$.

From $y'(L) = 0$: $\frac{w_0}{6}L^3 + \frac{C_1}{2}L^2 + C_2 L = 0 \quad \cdots$ (i)

From $y(L) = 0$: $\frac{w_0}{24}L^4 + \frac{C_1}{6}L^3 + \frac{C_2}{2}L^2 = 0 \quad \cdots$ (ii)

Solving equations (i) and (ii): $C_1 = -\frac{w_0 L}{2}$, $C_2 = \frac{w_0 L^2}{12}$.

$$\boxed{y(x) = \frac{w_0 x^2 (x - L)^2}{24EI}}$$

Maximum deflection at $x = L/2$:

$$y\left(\frac{L}{2}\right) = \frac{w_0 (L/2)^2 (L/2 - L)^2}{24EI} = \frac{w_0 \cdot (L^2/4) \cdot (L^2/4)}{24EI} = \frac{w_0 L^4}{384EI}$$

Comparison with simply supported beam:

$$\text{Simply supported: } |y_{\max}| = \frac{5 w_0 L^4}{384EI} \qquad \text{Fixed-fixed: } |y_{\max}| = \frac{w_0 L^4}{384EI}$$

The fixed-fixed beam is **5 times stiffer** — the clamped ends provide enormous additional restraint.

■

Example 5.5.4 — Numerical Beam Design

A simply supported steel floor beam spans $L = 8$ m and carries $w_0 = 20$ kN/m. $E = 200$ GPa. The allowable midspan deflection is $L/360$. Find the minimum required second moment of area I.

Solution:

Maximum deflection formula: $|y_{\max}| = \frac{5 w_0 L^4}{384EI}$

Allowable deflection: $\Delta_{\text{allow}} = \frac{L}{360} = \frac{8000}{360} = 22.22 \text{ mm} = 0.02222 \text{ m}$

Setting $|y_{\max}| \le \Delta_{\text{allow}}$:

$$\frac{5 \times 20{,}000 \times 8^4}{384 \times 200 \times 10^9 \times I} \le 0.02222$$

Numerator: $5 \times 20{,}000 \times 4096 = 409{,}600{,}000 \text{ N·m}^3$

Denominator factor: $384 \times 200 \times 10^9 = 7.68 \times 10^{13}$

$$I \ge \frac{409{,}600{,}000}{7.68 \times 10^{13} \times 0.02222} = \frac{409{,}600{,}000}{1{,}706{,}534{,}400} \ge 2.40 \times 10^{-4} \text{ m}^4 = 240 \times 10^6 \text{ mm}^4$$

Minimum required $I = 2.40 \times 10^{-4}$ **m**^{4} (a standard UB 406\$178\$74 section with $I \approx 274 \times 10^6$ mm^4 would suffice).

■

Exercises 5.5.1 — Beam Deflection

5.5.1 A simply supported beam, $L = 5$ m, $EI = 6 \times 10^6$ N·m^2, carries $w_0 = 18$ kN/m. Find: (a) deflection equation; (b) midspan deflection; (c) slope at both supports.

5.5.2 A cantilever beam, $L = 3$ m, carries $w_0 = 12$ kN/m (uniform, downward). $EI = 4 \times 10^6$ N·m^2. Derive the deflection equation and find: (a) maximum deflection at free end; (b) slope at the free end.

5.5.3 A simply supported beam carries a central point load $P = 40$ kN at $x = L/2$. $L = 6$ m, $EI = 10 \times 10^6$ N·m^2. Find the maximum deflection. [Answer: $PL^3/(48EI)$]

5.5.4 Show that the bending moment in a simply supported beam with uniform load w_0 is $M(x) = EI \cdot y''(x) = w_0 x(L - x)/2$. Verify it is maximum at $x = L/2$.

5.5.5 A propped cantilever is fixed at $x = 0$ and simply supported at $x = L$ with uniform w_0. State the four boundary conditions and set up (but do not fully solve) the system.

5.5.6 *(Design)* A timber beam ($E = 12$ GPa) spans $L = 5$ m, simply supported, carrying 15 kN/m. Allowable deflection $= L/300$. Find minimum I required.

5.5.2 Vibrations and Oscillations

5.5.2.1 Fundamentals of Oscillatory Systems

Mechanical vibrations are encountered in virtually every branch of engineering — from the seismic response of buildings, to the dynamic behaviour of rotating machinery, to the flutter of aircraft wings. The mathematical description of vibrations relies fundamentally on second and higher-order derivatives. The general governing equation of a single-degree-of-freedom (SDOF) mass-spring-damper system encapsulates the interplay of inertia (second derivative), damping (first derivative), and stiffness (zeroth derivative).

5.5.2.2 Undamped Free Vibration — Derivation of the Equation of Motion

Consider a mass m (kg) on a frictionless surface attached to a spring of stiffness k (N/m). Let x denote displacement from the equilibrium position. The spring exerts a restoring force $F_s = -kx$. Applying Newton's second law:

$$\sum F = ma \Longrightarrow -kx = m\ddot{x}$$

Rearranging to standard form:

$$m\ddot{x} + kx = 0$$

Dividing by m and defining the natural frequency $\omega_n = \sqrt{k/m}$:

$$\boxed{\ddot{x} + \omega_n^2 x = 0} \qquad \text{(Undamped SHM equation)}$$

The general solution is:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t) = C\cos(\omega_n t - \varphi)$$

where A, B (or equivalently C and φ) are determined by initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$A = x_0, \qquad B = \frac{v_0}{\omega_n}, \qquad C = \sqrt{x_0^2 + \frac{v_0^2}{\omega_n^2}}$$

Key relationships:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} \qquad T = \frac{2\pi}{\omega_n} \text{ (natural period, s)} \qquad f_n = \frac{\omega_n}{2\pi} \text{ (natural frequency, Hz)}$$

The velocity and acceleration are:

$$\dot{x}(t) = -A\omega_n\sin(\omega_n t) + B\omega_n\cos(\omega_n t)$$

$$\ddot{x}(t) = -A\omega_n^2\cos(\omega_n t) - B\omega_n^2\sin(\omega_n t) = -\omega_n^2\, x(t)$$

The relation $\ddot{x} = -\omega_n^2 x$ verifies that the solution satisfies the equation of motion and shows that the restoring acceleration is always proportional to — and opposite in direction to — the displacement.

5.5.2.3 Damped Free Vibration

In practice, all vibrating systems dissipate energy through friction, internal damping in materials, or fluid resistance. For a linear viscous damper (dashpot) with damping coefficient c (N·s/m), the damping force is $F_d = -c\dot{x}$. Newton's second law gives:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Dividing by m:

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

where two fundamental parameters are defined:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{(undamped natural frequency, rad/s)}$$

$$\zeta = \frac{c}{2\sqrt{km}} \qquad \text{(damping ratio, dimensionless)}$$

The critical damping coefficient is $c_c = 2\sqrt{km} = 2m\omega_n$, and $\zeta = c/c_c$.

The characteristic equation (assuming $x = e^{st}$):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \Longrightarrow s = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Three physically distinct cases arise:

Case 1: Underdamped ($\zeta < 1$)

Complex conjugate roots: $s = -\zeta\omega_n \pm i\omega_d$, where the damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

$$x(t) = e^{-\zeta\omega_n t}[A\cos(\omega_d t) + B\sin(\omega_d t)] = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \varphi)$$

The motion is oscillatory with exponentially decaying amplitude. The envelope of the oscillation is $Ce^{-\zeta\omega_n t}$. This is the most common situation in practice (most structures and machines are underdamped).

Case 2: Critically Damped ($\zeta = 1$)

Repeated real root: $s = -\omega_n$ (double root).

$$x(t) = (A + Bt)\, e^{-\omega_n t}$$

The system returns to equilibrium as rapidly as possible without oscillating. Used in precision instruments, electrical meter movements, and hydraulic door closers.

Case 3: Overdamped ($\zeta > 1$)

Two distinct negative real roots: $s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$.

$$x(t) = Ae^{s_1 t} + Be^{s_2 t}$$

The system decays exponentially without oscillation, but more slowly than the critically damped case.

5.5.2.4 Forced Vibration with Harmonic Excitation

When an external harmonic force $F(t) = F_0\cos(\Omega t)$ is applied:

$$m\ddot{x} + c\dot{x} + kx = F_0\cos(\Omega t)$$

The complete solution has two parts: $x(t) = x_h(t) + x_p(t)$, where x_h is the transient (homogeneous) solution that decays with time, and x_p is the steady-state (particular) solution:

$$x_p(t) = X\cos(\Omega t - \varphi)$$

where the steady-state amplitude X and phase φ are:

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \varphi = \arctan\left[\frac{2\zeta r}{1-r^2}\right]$$

and $r = \Omega/\omega_n$ is the frequency ratio. The Dynamic Magnification Factor (DMF) is:

$$\text{DMF} = \frac{X}{F_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

At resonance ($r \to 1$): DMF $\to 1/(2\zeta)$, which approaches infinity as $\zeta \to 0$. This is why resonance is catastrophic in undamped systems. Even modest levels of damping ($\zeta = 0.05$) limit the amplitude to DMF $= 10$ at resonance.

5.5.2.5 Logarithmic Decrement

For underdamped free vibration, the damping ratio can be measured experimentally using successive amplitude peaks. If x_1 and x_2 are the amplitudes of two successive peaks (separated by one damped period $T_d = 2\pi/\omega_d$):

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

This is the **logarithmic decrement**. For small ζ, $\delta \approx 2\pi\zeta$. This formula is widely used in experimental vibration testing to extract damping ratios from measured decay curves.

Worked Examples — Section 5.5.2

Example 5.5.5 — Undamped Mass-Spring System

A machine component of mass $m = 8$ kg is mounted on a spring of stiffness $k = 3200$ N/m. Initial conditions: $x(0) = 0.05$ m, $\dot{x}(0) = 0.4$ m/s.

116. Find ω_n, period T, and natural frequency f_n. (b) Write the complete solution $x(t)$. (c) Find the amplitude C. (d) Find velocity and acceleration at $t = 0.1$ s. (e) Find maximum acceleration and the time it first occurs.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3200}{8}} = \sqrt{400} = 20 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0.3142 \text{ s} \qquad f_n = \frac{20}{2\pi} \approx 3.183 \text{ Hz}$$

117. $x(t) = A\cos(20t) + B\sin(20t)$

$x(0) = A = 0.05$ m; $\quad \dot{x}(0) = 20B = 0.4 \Longrightarrow B = 0.02$ m.

$$x(t) = 0.05\cos(20t) + 0.02\sin(20t) \text{ m}$$

118. $C = \sqrt{A^2 + B^2} = \sqrt{0.0025 + 0.0004} = \sqrt{0.0029} \approx 0.05385$ m ≈ 53.85 mm.

119. At $t = 0.1$ s:

$$\dot{x}(t) = -\sin(20t) + 0.4\cos(20t)$$

$$\dot{x}(0.1) = -\sin 2 + 0.4\cos 2 = -0.9093 + 0.4(-0.4161) = -1.076 \text{ m/s}$$

$$\ddot{x}(t) = -\omega_n^2\, x(t) = -400\, x(t)$$

$$x(0.1) = 0.05\cos 2 + 0.02\sin 2 = 0.05(-0.4161) + 0.02(0.9093) = -0.00262 \text{ m}$$

$$\ddot{x}(0.1) = -400(-0.00262) = 1.048 \text{ m/s}^2$$

120. Maximum acceleration: $|\ddot{x}|_{\max} = \omega_n^2 C = 400 \times 0.05385 = 21.54$ m/s^2.

First occurrence when $x(t) = -C$ (most negative), i.e. $\cos(20t - \varphi) = -1$, where $\varphi = \arctan(B/A) = \arctan(0.4) \approx 0.3805$ rad:

$$20t - 0.3805 = \pi \Longrightarrow t = \frac{\pi + 0.3805}{20} \approx 0.1760 \text{ s}$$

■

Example 5.5.6 — Damped Vibration: Vehicle Suspension

A vehicle suspension has: $m = 350$ kg, $k = 35{,}000$ N/m, $c = 3500$ N·s/m.

Initial: $x(0) = 0.06$ m, $\dot{x}(0) = 0$.

121. Find ω_n, ζ, ω_d. Classify the system. (b) Write the complete response $x(t)$. (c) Find the first peak amplitude and time to first peak. (d) Calculate logarithmic decrement and ratio of successive peak amplitudes.

Solution:

$$\omega_n = \sqrt{\frac{35{,}000}{350}} = \sqrt{100} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{3500}{2 \times 350 \times 10} = 0.5 \qquad \omega_d = 10\sqrt{1 - 0.25} = 10\sqrt{0.75} = 8.660 \text{ rad/s}$$

$\zeta = 0.5 < 1 \rightarrow$ **UNDERDAMPED**.

122. $x(t) = e^{-5t}[A\cos(8.660t) + B\sin(8.660t)]$

$x(0) = A = 0.06$ m.

$\dot{x}(0) = -5A + B\omega_d = 0 \Longrightarrow B = \frac{5 \times 0.06}{8.660} = 0.03464$ m.

$$x(t) = e^{-5t}[0.06\cos(8.660t) + 0.03464\sin(8.660t)] \text{ m}$$

123. First peak when $\dot{x} = 0$ (after $t = 0$):

$$\tan(\omega_d t) = \frac{\omega_d}{\zeta\omega_n} = \frac{8.660}{5} = 1.7321 \Longrightarrow \omega_d t = \frac{\pi}{3} \Longrightarrow t_{\text{peak}} = \frac{\pi/3}{8.660} \approx 0.1209 \text{ s}$$

$$x(t_{\text{peak}}) = e^{-5\times0.1209}[0.06\cos(\pi/3) + 0.03464\sin(\pi/3)] = e^{-0.6046}(0.03 + 0.03000)$$
$$= 0.5464 \times 0.06000 = 0.03278 \text{ m}$$

124. $T_d = 2\pi/\omega_d = 0.7255$ s.

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.5)}{\sqrt{0.75}} = \frac{\pi}{0.8660} = 3.628$$

$$\frac{x_2}{x_1} = e^{-\delta} = e^{-3.628} = 0.02664$$

Each peak is only 2.66% of the previous — the suspension damps very rapidly.

■

Example 5.5.7 — Resonance in Forced Vibration

A machine ($m = 120$ kg) is mounted on a spring-damper isolator: $k = 120{,}000$ N/m, $c = 960$ N·s/m. Unbalance force: $F(t) = 800\cos(\Omega t)$ N, with $\Omega = 31.4$ rad/s (≈ 300 RPM).

125. Find ω_n and ζ. (b) Find the frequency ratio r. (c) Calculate steady-state amplitude X. (d) Determine phase angle φ. (e) Compare X with static deflection and comment.

Solution:

$$\omega_n = \sqrt{\frac{120{,}000}{120}} = \sqrt{1000} = 31.62 \text{ rad/s} \qquad \zeta = \frac{960}{2 \times 120 \times 31.62} = 0.1265$$

126. $r = \Omega/\omega_n = 31.4/31.62 = 0.9930$ (very close to resonance).

$$1 - r^2 = 1 - 0.9860 = 0.0140 \qquad 2\zeta r = 2(0.1265)(0.9930) = 0.2512$$

$$\text{DMF} = \frac{1}{\sqrt{(0.0140)^2 + (0.2512)^2}} = \frac{1}{\sqrt{0.06330}} = 3.975$$

Static deflection: $\delta_{st} = F_0/k = 800/120{,}000 = 0.006667$ m.

$$X = \text{DMF} \times \delta_{st} = 3.975 \times 0.006667 = 0.02650 \text{ m} = 26.50 \text{ mm}$$

127. $\varphi = \arctan\left[\frac{0.2512}{0.0140}\right] = \arctan(17.94) \approx 86.8°$.

The response nearly lags the excitation by 90° — a hallmark of near-resonance.

128. Static deflection = 6.67 mm; Dynamic amplitude = 26.50 mm — amplification factor ≈ 3.97 ×. Despite meaningful damping ($\zeta = 12.65\%$), near-resonance still amplifies response by approximately 4 times. In practice, the machine operating speed should be shifted away from ω_n.

■

Example 5.5.8 — Torsional Vibration of an Engine Shaft

A circular steel shaft: $L = 1.2$ m, $d = 50$ mm. A flywheel with polar mass moment of inertia $J_d = 0.8$ kg·m² is attached at the free end; the other end is fixed. $G = 80$ GPa (shear modulus for steel).

129. Find the polar second moment of area. (b) Find torsional stiffness k_t. (c) Set up and solve the free torsional vibration equation. (d) Find natural frequency.

Solution:

$$J_{\text{shaft}} = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 6.136 \times 10^{-7} \text{ m}^4$$

$$k_t = \frac{G J_{\text{shaft}}}{L} = \frac{80 \times 10^9 \times 6.136 \times 10^{-7}}{1.2} = 40{,}907 \text{ N·m/rad}$$

130. Equation of motion for torsional vibration:

$$J_d \ddot{\theta} + k_t \theta = 0 \Longrightarrow \ddot{\theta} + \frac{k_t}{J_d}\theta = 0$$

This is the exact torsional analogue of the linear SHM equation $\ddot{x} + \omega_n^2 x = 0$.

General solution: $\theta(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$.

$$\omega_n = \sqrt{\frac{k_t}{J_d}} = \sqrt{\frac{40{,}907}{0.8}} = \sqrt{51{,}134} \approx 226.1 \text{ rad/s}$$

$$f_n = \frac{226.1}{2\pi} \approx 35.98 \text{ Hz} \approx 36 \text{ Hz} \qquad T = 1/f_n \approx 0.0278 \text{ s}$$

Engineering note: This frequency is well above typical engine excitation frequencies (engine at 3000 RPM produces 2nd-order excitation at 100 Hz), so resonance is not an issue in normal operating range.

■

Exercises 5.5.2 — Vibrations and Oscillations

5.5.6 A 12 kg mass on a spring ($k = 4800$ N/m) is displaced 40 mm and released with initial velocity 0.6 m/s. Find: (a) ω_n and T; (b) $x(t)$; (c) amplitude; (d) maximum velocity and maximum acceleration.

5.5.7 A structure has $m = 5000$ kg, $k = 2{,}000{,}000$ N/m, $c = 40{,}000$ N·s/m. (a) Find ω_n, ζ, and classify; (b) find ω_d; (c) write $x(t)$ for $x_0 = 0.03$ m, $v_0 = 0$.

5.5.8 Verify that $x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \varphi)$ satisfies $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ by direct substitution.

5.5.9 A machine operates at 1500 RPM. Mounting: $m = 80$ kg, $k = 200{,}000$ N/m, $\zeta = 0.07$. Unbalance force $F_0 = 600$ N. Find: (a) r; (b) DMF; (c) steady-state amplitude.

5.5.10 An undamped spring-mass system must resonate at exactly 8 Hz. If $m = 5$ kg, find: (a) required k; (b) amplitude if $x_0 = 0$, $v_0 = 0.5$ m/s; (c) jerk at $t = 0$.

5.5.11 For an underdamped system with $\delta = 0.80$ (logarithmic decrement), find ζ and the ratio of amplitudes after 5 complete oscillations.

Chapter 5 Summary

Section	Key Results and Formulas
5.1 Higher Derivatives	$(e^{ax})^{(n)} = a^n e^{ax}$ | $(\sin ax)^{(n)} = a^n \sin(ax + n\pi/2)$ | $(\ln x)^{(n)} = (-1)^{n-1}(n-1)!/x^n$
5.2 Physical Meaning	s' = velocity; s'' = acceleration; s''' = jerk. Speeding up ⇔ v and a same sign.
5.3 Concavity	$f'' > 0$: concave up; $f'' < 0$: concave down. Inflection where f'' changes sign.
5.3 Second Derivative Test	$f'(c) = 0$, $f''(c) > 0 \rightarrow$ local min; $f''(c) < 0 \rightarrow$ local max; $f''(c) = 0 \rightarrow$ inconclusive.
5.4 Leibniz Rule	$(fg)^{(n)} = \sum_k \binom{n}{k} f^{(k)} g^{(n-k)}$ — mirrors the Binomial Theorem.
5.5 Beam Deflection	$EI\, y^{(4)} = w(x)$. Simply supported + UDL: $\lvert y_{\max}\rvert = 5w_0L^4/(384EI)$.
5.5 Vibrations	$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$; $\zeta < 1$ underdamped; DMF $= 1/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$.

Chapter Review Problems

R1. Find the nth derivative of $f(x) = (2x + 3)^5$ without expanding.

R2. A particle's position is $s(t) = 2t^4 - 12t^3 + 18t^2$. Find intervals when: (a) particle moves in the positive direction; (b) speed is increasing.

R3. Find all inflection points of $f(x) = x^4 - 4x^3 + 6x^2$ and verify by sign analysis.

R4. Use the Leibniz rule to find $h^{(4)}(x)$ for $h(x) = x^3 e^{-x}$.

R5. A cantilever beam ($L = 3$ m, $EI = 5 \times 10^6$ N·m^2) carries 15 kN/m uniform load. Find maximum deflection and slope at the free end.

R6. A 60 kg mass is attached to a spring ($k = 24{,}000$ N/m) with $c = 720$ N·s/m. Find ω_n, ζ, ω_d, and write $x(t)$ for $x_0 = 0.03$ m, $v_0 = 0$.

R7. Derive from first principles the nth derivative of $f(x) = 1/(1 + x)$.

R8. A machine runs at 1200 RPM on a mounting with $m = 60$ kg, $k = 400{,}000$ N/m, $\zeta = 0.05$. Unbalance force $F_0 = 300$ N. Find the steady-state amplitude and comment on resonance risk.

R9. Prove that the fixed-fixed beam under UDL has exactly 5 times greater stiffness than the equivalent simply supported beam (compare maximum deflection formulas).

R10. Find $h^{(n)}(x)$ for $h(x) = e^{2x}\cos(3x)$. Express using $r = \sqrt{13}$ and $\theta = \arctan(3/2)$.

Selected Solutions to Exercises

Exercise 5.1.1

$$f(x) = 3x^7 - 2x^5 + x^3 - 4x + 9$$

$$f'(x) = 21x^6 - 10x^4 + 3x^2 - 4$$

$$f''(x) = 126x^5 - 40x^3 + 6x$$

$$f'''(x) = 630x^4 - 120x^2 + 6$$

$$f^{(4)}(x) = 2520x^3 - 240x$$

$$f^{(5)}(x) = 7560x^2 - 240$$

Exercise 5.2.1

$$h(t) = 80t^2 - 8/3\, t^3$$

131. $h'(t) = 160t - 8t^2$; $h'(4) = 640 - 128 = 512$ m/s.

132. $h''(t) = 160 - 16t$; $h''(6) = 160 - 96 = 64$ m/s^2.

133. $h''(t) = 0 \Rightarrow t = 10$ s; maximum velocity at $t = 10$: $h'(10) = 1600 - 800 = 800$ m/s.

134. $j(t) = h'''(t) = -16$ m/s^3 (constant — constant jerk implies linearly changing acceleration).

Exercise 5.3.1(a)

$f(x) = x^3 - 3x^2 + 4$; $f''(x) = 6x - 6 = 6(x - 1)$.

$f'' < 0$ for $x < 1$ (concave down); $f'' > 0$ for $x > 1$ (concave up).

f'' changes sign at $x = 1$: inflection point at $\big(1,\ f(1)\big) = (1,\ 2)$.

Exercise 5.4.1(a)

$h(x) = x^2 e^x$. $f = x^2$, $g = e^x$; $f' = 2x$, $f'' = 2$, $f^{(k)} = 0$ for $k \geq 3$:

$$h^{(n)}(x) = \binom{n}{0} x^2 e^x + \binom{n}{1} 2x\, e^x + \binom{n}{2} 2\, e^x = e^x\big(x^2 + 2nx + n(n-1)\big)$$

Exercise 5.5.3

Simply supported beam, central point load P, span L. The deflection equation (from integration with appropriate BCs) gives:

$$|y_{\max}| = \frac{PL^3}{48EI}$$

Numerically: $P = 40{,}000$ N, $L = 6$ m, $EI = 10 \times 10^6$ N·m^2:

$$|y_{\max}| = \frac{40{,}000 \times 216}{48 \times 10{,}000{,}000} = \frac{8{,}640{,}000}{480{,}000{,}000} = 0.018 \text{ m} = 18 \text{ mm}$$

Exercise 5.5.6

$m = 12$ kg, $k = 4800$ N/m, $x_0 = 0.04$ m, $v_0 = 0.6$ m/s.

$$\omega_n = \sqrt{4800/12} = 20 \text{ rad/s} \qquad T = \pi/10 \approx 0.3142 \text{ s}$$

$A = x_0 = 0.04$ m; $B = v_0/\omega_n = 0.6/20 = 0.03$ m.

$$x(t) = 0.04\cos(20t) + 0.03\sin(20t) \text{ m}$$

$$C = \sqrt{0.04^2 + 0.03^2} = \sqrt{0.0025} = 0.05 \text{ m}$$

$$v_{\max} = C\omega_n = 0.05 \times 20 = 1.0 \text{ m/s} \qquad a_{\max} = C\omega_n^2 = 0.05 \times 400 = 20 \text{ m/s}^2$$

End of Chapter 5. This chapter has derived every result from first principles and demonstrated each with engineering-context worked examples. Solutions to all exercises are available in the Solutions Manual.

Chapter 6 — Higher-Order Derivatives

CHAPTER 6

Higher-Order Derivatives

nth Derivatives • Concavity • Leibniz Rule • Beam Deflection • Vibrations

Higher-order derivatives arise naturally whenever we differentiate a function more than once. While the first derivative captures the instantaneous rate of change of a function, second and higher derivatives reveal deeper geometric, physical, and structural information that is indispensable in engineering analysis, mathematical physics, and applied science. The curvature of a beam under load, the acceleration of a vibrating machine, the shape of a profit curve, and the oscillatory decay of a damped spring are all governed by relationships that involve second, third, and fourth derivatives.

This chapter develops the theory of higher-order derivatives in a rigorous and systematic way. We begin with formal definitions and derive closed-form formulas for the nth derivative of all standard elementary functions — exponentials, trigonometric functions, logarithms, and power functions. We then explore the profound physical interpretations of the derivative hierarchy: velocity, acceleration, jerk, snap, and beyond. Section 6.3 applies the second derivative to the geometry of curves, establishing the theory of concavity and inflection points alongside the second derivative test for classifying critical points. Section 6.4 derives the powerful Leibniz product rule for the nth derivative — one of the most elegant results in differential calculus — and demonstrates its application through numerous worked examples. Finally, Section 6.5 applies the full theoretical machinery to two central areas of engineering: the deflection of beams governed by a fourth-order differential equation, and the analysis of mechanical vibrations described by second-order equations of motion. Throughout, derivations are carried out from first principles, and every result is verified and illustrated through original worked examples.

6.1 Second and Higher Derivatives

6.1.1 Definitions and Notation

Let f be a function that is differentiable on an open interval I. The first derivative $f'(x)$ is itself a function on I (or a subset thereof). If $f'(x)$ is also differentiable, we may differentiate it again to obtain the second derivative:

$$f''(x) = \frac{d}{dx}[f'(x)]$$

More generally, the nth derivative of f is defined recursively as:

$$f^{(n)}(x) = \frac{d}{dx}\left[f^{(n-1)}(x)\right], \qquad n = 1,2,3,\ldots$$

with the convention $f^{(0)}(x) = f(x)$. Three major notational systems are used throughout mathematics and engineering:

135. **Lagrange (prime) notation:** $f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x)$

136. **Leibniz notation:** $\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots,\ \frac{d^ny}{dx^n}$

137. **Newton dot notation (time derivatives):** $\dot{y},\ \ddot{y}$ (rarely used beyond the second)

In Leibniz notation, the second derivative is formally written as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

This notation is suggestive of algebraic manipulation but should be understood as an operator acting on f — not a ratio of infinitesimals. The Leibniz notation is preferred in engineering because it makes the variable of differentiation explicit, which is critical in multi-variable settings and differential equations.

For a function $y = f(x)$, the relationship between notations is:

$$f^{(2)}(x) = f''(x) = \frac{d^2y}{dx^2} \qquad \text{(second derivative)}$$

$$f^{(3)}(x) = f'''(x) = \frac{d^3y}{dx^3} \qquad \text{(third derivative)}$$

$$f^{(n)}(x) = \frac{d^ny}{dx^n} \qquad (n\text{th derivative})$$

6.1.2 Derivatives of Power Functions

Let $f(x) = x^m$ where m is a positive integer. Successive differentiation yields:

$$f'(x) = m\,x^{m-1}$$

$$f''(x) = m(m-1)\,x^{m-2}$$

$$f'''(x) = m(m-1)(m-2)\,x^{m-3}$$

$$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)\,x^{m-k} \qquad \text{for } k \le m$$

$$f^{(m)}(x) = m! \qquad \text{(constant)}$$

$$f^{(k)}(x) = 0 \qquad \text{for all } k > m$$

Using the falling factorial notation $m^{\underline{k}} = m!/(m-k)!$, the general formula becomes:

$$f^{(k)}(x) = \frac{m!}{(m-k)!}\,x^{m-k} \qquad (k \le m)$$

6.1.3 Derivation: nth Derivative of e^{ax}

Let $f(x) = e^{ax}$. We claim that $f^{(n)}(x) = a^n e^{ax}$ for all $n \ge 1$.

Proof by Mathematical Induction:

Base case ($n = 1$): $f'(x) = ae^{ax} = a^1 e^{ax}$. ✓

Inductive step: Assume $f^{(k)}(x) = a^k e^{ax}$ for some $k \ge 1$. Then:

$$f^{(k+1)}(x) = \frac{d}{dx}[a^k e^{ax}] = a^k \cdot a\,e^{ax} = a^{k+1}e^{ax} \quad ✓$$

Therefore, by induction, the formula holds for all $n \ge 1$.

$$\boxed{(e^{ax})^{(n)} = a^n e^{ax}}$$

6.1.4 Derivation: nth Derivative of $\sin(ax)$ and $\cos(ax)$

Let $f(x) = \sin(ax)$. The first four derivatives follow a repeating cycle of period 4:

$$f'(x) = a\cos(ax)$$

$$f''(x) = -a^2\sin(ax)$$

$$f'''(x) = -a^3\cos(ax)$$

$$f^{(4)}(x) = a^4\sin(ax) \qquad \leftarrow \text{identical to } f(x) \text{ scaled by } a^4$$

We claim the general formula is:

$$\boxed{(\sin ax)^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)}$$

Proof by Mathematical Induction:

Base case ($n = 1$): $a^1 \sin(ax + \pi/2) = a\cos(ax)$. ✓ (since $\sin(\theta + \pi/2) = \cos\theta$)

Inductive step: Assume $f^{(k)}(x) = a^k \sin(ax + k\pi/2)$. Differentiating:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[a^k \sin\left(ax + \frac{k\pi}{2}\right)\right] = a^k \cdot a\cos\left(ax + \frac{k\pi}{2}\right)$$

$$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right) = a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right) \quad \checkmark$$

Similarly, for $f(x) = \cos(ax)$:

$$\boxed{(\cos ax)^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)}$$

Verification ($n = 1$): $a\cos(ax + \pi/2) = a(-\sin ax) = -a\sin(ax)$. Since $\cos(\theta + \pi/2) = -\sin\theta$, this gives $(\cos ax)' = -a\sin(ax)$. ✓

6.1.5 Derivation: nth Derivative of $\ln x$

Let $f(x) = \ln x$, $x > 0$. Computing successive derivatives:

$$f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -6x^{-4}$$

The general pattern is:

$$\boxed{(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}} \qquad (n \geq 1)$$

Proof — Inductive step: Assume the formula holds for k. Then:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[(-1)^{k-1}(k-1)!\, x^{-k}\right] = (-1)^{k-1}(k-1)! \cdot (-k)\, x^{-k-1}$$

$$= (-1)^k\, k!\, x^{-(k+1)} \quad \checkmark$$

6.1.6 Reference Table: nth Derivatives of Common Functions

$f(x)$	$f^{(n)}(x)$	Conditions
x^m	$\frac{m!}{(m-n)!}\, x^{m-n}$	$n \leq m$; equals $m!$ when $n = m$; 0 when $n > m$
e^x	e^x	All n
e^{ax}	$a^n e^{ax}$	All n, a constant

$f(x)$	$f^{(n)}(x)$	Conditions
$\sin(ax)$	$a^n \sin(ax + n\pi/2)$	All n
$\cos(ax)$	$a^n \cos(ax + n\pi/2)$	All n
$\ln x$	$(-1)^{n-1}(n-1)!/x^n$	$n \geq 1, x > 0$
$(1+x)^m$	$m(m-1)\cdots(m-n+1)(1+x)^{m-n}$	Any real m
$1/(1-x)$	$n!/(1-x)^{n+1}$	$\lvert x \rvert < 1$

Worked Examples — Section 6.1

Example 6.1.1 — Polynomial Derivatives

Find the first six derivatives of $f(x) = 2x^6 - 5x^4 + 3x^2 - 7x + 4$.

Solution:

$$f'(x) = 12x^5 - 20x^3 + 6x - 7$$

$$f''(x) = 60x^4 - 60x^2 + 6$$

$$f'''(x) = 240x^3 - 120x$$

$$f^{(4)}(x) = 720x^2 - 120$$

$$f^{(5)}(x) = 1440x$$

$$f^{(6)}(x) = 1440 \qquad \text{(constant)}$$

$f^{(k)}(x) = 0$ for all $k \geq 7$ (a degree-6 polynomial vanishes after 6 differentiations).

■

Example 6.1.2 — nth Derivative of $e^x \sin x$

Find the general formula for $f^{(n)}(x)$ when $f(x) = e^x \sin x$.

Solution — using the complex representation method:

Write $e^x \sin x = \text{Im}\left(e^x \cdot e^{ix}\right) = \text{Im}\left(e^{(1+i)x}\right)$.

The nth derivative of $e^{(1+i)x}$ is $(1+i)^n e^{(1+i)x}$.

Convert $(1+i)$ to polar form: $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\theta = \arctan(1/1) = \pi/4$.

So $(1+i)^n = \left(\sqrt{2}\right)^n e^{in\pi/4} = 2^{n/2}\left(\cos(n\pi/4) + i\sin(n\pi/4)\right)$.

Therefore:

$$(e^x \sin x)^{(n)} = \text{Im}\left[2^{n/2}\, e^{(1+i)x}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right]$$

$$= 2^{n/2}\, e^x \text{Im}\left[e^{ix}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right]$$

$$= \boxed{2^{n/2}\, e^x \sin\left(x + \frac{n\pi}{4}\right)}$$

Verification ($n = 1$): $\sqrt{2}\, e^x \sin(x + \pi/4) = \sqrt{2}\, e^x \left[\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right] = e^x(\sin x + \cos x)$.

Direct check: $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$. ✓

■

Example 6.1.3 — $f^{(4)}(x)$ for $x^3 \ln x$

Find $f^{(4)}(x)$ for $f(x) = x^3 \ln x$.

Solution — Successive differentiation:

$$f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x} = 3x^2 \ln x + x^2$$

$$f''(x) = 6x \ln x + 3x^2 \cdot \frac{1}{x} + 2x = 6x \ln x + 3x + 2x = 6x \ln x + 5x$$

$$f'''(x) = 6 \ln x + 6x \cdot \frac{1}{x} + 5 = 6 \ln x + 11$$

$$f^{(4)}(x) = \frac{6}{x}$$

Note: All higher derivatives follow: $f^{(5)}(x) = -6/x^2$, $f^{(6)}(x) = 12/x^3$, ...

■

Exercises 6.1

6.1.1 Find $f^{(5)}(x)$ for $f(x) = 3x^7 - 2x^5 + x^3 - 4x + 9$.

6.1.2 Find $f^{(n)}(x)$ for each: (a) $f(x) = e^{-3x}$ (b) $f(x) = \sin(4x)$ (c) $f(x) = \cos(x/3)$ (d) $f(x) = \ln(2x)$

6.1.3 Find $f^{(4)}(x)$ for $f(x) = x^2 e^x$ using successive differentiation.

6.1.4 Derive the general formula for the nth derivative of $f(x) = 1/(1-x)$, $|x| < 1$. [*Hint:* Write as $(1-x)^{-1}$ and differentiate repeatedly. Verify against the geometric series.]

6.1.5 Find the nth derivative of $f(x) = xe^x$. [Answer: $f^{(n)}(x) = (x + n)e^x$]

6.1.6 Prove by induction that $(\cos ax)^{(n)} = a^n \cos(ax + n\pi/2)$.

6.2 Physical Meaning: Acceleration, Jerk, and Beyond

6.2.1 The Kinematic Hierarchy of Derivatives

In classical mechanics, the motion of a particle along a straight line is completely described by its position function $s(t)$, giving displacement from a reference point as a function of time t. Each successive derivative of $s(t)$ carries a distinct, physically meaningful interpretation, forming a natural hierarchy:

Derivative	Symbol	Name	SI Unit	Physical Meaning
$s(t)$	s	Position / Displacement	m	Location relative to reference
$s'(t) = ds/dt$	$v(t)$	Velocity	m/s	Rate of change of position
$s''(t) = d^2s/dt^2$	$a(t)$	Acceleration	m/s^2	Rate of change of velocity
$s'''(t) = d^3s/dt^3$	$j(t)$	Jerk (Jolt)	m/s^3	Rate of change of acceleration
$s^{(4)}(t) = d^4s/dt^4$	snap(t)	Snap (Jounce)	m/s^4	Rate of change of jerk
$s^{(5)}(t)$	crackle(t)	Crackle	m/s^5	Rate of change of snap
$s^{(6)}(t)$	pop(t)	Pop	m/s^6	Rate of change of crackle

Velocity $v(t) = s'(t)$ measures how rapidly the particle's position is changing and can be positive, negative, or zero. Acceleration $a(t) = s''(t) = v'(t)$ measures how the velocity is changing — it is directly related to force by Newton's second law: $F = ma$. Jerk $j(t) = s'''(t)$ is the third derivative and quantifies abruptness of motion. Jerk is a critical engineering parameter in the design of elevators, railway systems, robotic arms, and vehicles, as humans are sensitive to sudden changes in acceleration. High jerk creates mechanical stress and passenger discomfort.

The fourth derivative, snap or jounce, appears in precision trajectory planning in robotics and aerospace engineering. The fifth and sixth derivatives (crackle and pop) arise primarily in satellite attitude control and semiconductor manufacturing robots, where trajectories must satisfy higher-order continuity constraints to avoid inducing vibrations in ultra-sensitive systems.

6.2.2 Sign Interpretation and Speed

An important distinction exists between velocity and speed: speed is the magnitude of velocity ($|v(t)|$) and is always non-negative. A particle speeds up when its velocity and acceleration have the same sign, and slows down when they have opposite signs:

Condition	Physical Interpretation
$v(t) > 0$	Moving in the positive direction
$v(t) < 0$	Moving in the negative direction
$v(t) = 0$	Momentarily at rest
$a(t) > 0$	Velocity is increasing
$a(t) < 0$	Velocity is decreasing
$v(t)$ and $a(t)$ same sign	Speeding up (speed increasing)
$v(t)$ and $a(t)$ opposite sign	Slowing down (speed decreasing)

6.2.3 Simple Harmonic Motion and Its Derivatives

For a particle executing simple harmonic motion (SHM):

$$s(t) = A\sin(\omega t + \varphi)$$

where A is amplitude (m), ω is angular frequency (rad/s), and φ is the initial phase angle (rad). The full derivative hierarchy is:

$$v(t) = s'(t) = A\omega\cos(\omega t + \varphi)$$

$$a(t) = s''(t) = -A\omega^2\sin(\omega t + \varphi) = -\omega^2 s(t)$$

$$j(t) = s'''(t) = -A\omega^3\cos(\omega t + \varphi) = -\omega^2 v(t)$$

The key relation $a(t) = -\omega^2 s(t)$ is the defining differential equation of SHM, showing that the restoring acceleration is proportional to — and oppositely directed to — displacement. This will be the cornerstone of Section 6.5.2.

6.2.4 Free Fall Under Gravity

For an object launched vertically with initial velocity v_0 from height s_0, ignoring air resistance:

$$s(t) = s_0 + v_0 t - 1/2\, g t^2$$

where $g = 9.81$ m/s^2 is the acceleration due to gravity.

$$v(t) = v_0 - gt \qquad a(t) = -g \text{ (constant)} \qquad j(t) = 0$$

The zero jerk in free fall reflects the fact that gravity is a constant force. In contrast, rocket propulsion produces nonzero jerk because thrust varies with time.

Worked Examples — Section 6.2

Example 6.2.1 — Particle Motion Analysis

A particle moves along the x-axis with position:

$$s(t) = t^3 - 6t^2 + 9t + 2 \qquad \text{(metres, } t \text{ in seconds, } t \geq 0)$$

138. Find velocity, acceleration, and jerk functions. (b) Find when the particle is at rest. (c) Find when the particle is speeding up and slowing down. (d) Find the total distance travelled in the first 4 seconds.

Solution:

$$v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$$

$$a(t) = s''(t) = 6t - 12 = 6(t-2)$$

$$j(t) = s'''(t) = 6 \text{ m/s}^3 \quad \text{(constant)}$$

139. At rest: $v(t) = 0 \Longrightarrow 3(t-1)(t-3) = 0 \Longrightarrow t = 1$ s and $t = 3$ s.

140. Sign table for $v(t)$ and $a(t)$:

141. $(0,1)$: $v(0.5) = 3.75 > 0$; $a(0.5) = -9 < 0 \rightarrow$ **Slowing down**

142. $(1,2)$: $v(1.5) = -2.25 < 0$; $a(1.5) = -3 < 0 \rightarrow$ **Speeding up**

143. $(2,3)$: $v(2.5) = -2.25 < 0$; $a(2.5) = 3 > 0 \rightarrow$ **Slowing down**

144. $(3,4)$: $v(3.5) = 3.75 > 0$; $a(3.5) = 9 > 0 \rightarrow$ **Speeding up**

145. Positions at key times: $s(0) = 2$ m, $s(1) = 6$ m, $s(3) = 2$ m, $s(4) = 6$ m.

Total distance $= |s(1) - s(0)| + |s(3) - s(1)| + |s(4) - s(3)| = 4 + 4 + 4 = 12$ m

■

Example 6.2.2 — Engineering Vehicle Test

An engineering test vehicle's position is $s(t) = 60t - 3t^2 + 1/9\, t^3$ (m, $0 \leq t \leq 15$ s).

146. Find velocity, acceleration, and jerk. (b) Find when acceleration equals zero. (c) Find velocity at the instant of zero acceleration. (d) Calculate the jerk at $t = 6$ s and comment on its engineering significance.

Solution:

$$v(t) = 60 - 6t + 1/3\, t^2 \qquad a(t) = -6 + 2/3\, t \qquad j(t) = 2/3 \text{ m/s}^3 \text{ (constant)}$$

147. $a(t) = 0$: $-6 + 2/3\, t = 0 \Longrightarrow t = 9$ s.

148. $v(9) = 60 - 54 + 1/3\,(81) = 60 - 54 + 27 = 33$ m/s.

149. $j(6) = 2/3$ m/s^3 (same everywhere — constant jerk profile throughout the test).

Engineering significance: A constant jerk profile is often deliberately designed into vehicle acceleration systems (called a "jerk-limited" profile) because it produces smooth,

predictable acceleration changes that minimise passenger discomfort and mechanical wear on drivetrain components.

■

Example 6.2.3 — Damped Vibration Derivatives

The displacement of a shock absorber is $s(t) = e^{-2t}\sin(5t)$ metres.

Find the velocity, acceleration, and jerk. Evaluate jerk at $t = 0$.

Solution:

$$v(t) = s'(t) = -2e^{-2t}\sin(5t) + 5e^{-2t}\cos(5t) = e^{-2t}(-2\sin 5t + 5\cos 5t)$$

$$a(t) = v'(t) = \frac{d}{dt}[e^{-2t}(-2\sin 5t + 5\cos 5t)]$$

$$= -2e^{-2t}(-2\sin 5t + 5\cos 5t) + e^{-2t}(-10\sin 5t - 10\cos 5t)$$

$$= e^{-2t}[(4\sin 5t - 10\cos 5t) + (-10\sin 5t - 10\cos 5t)]$$

$$= e^{-2t}(-6\sin 5t - 20\cos 5t) = -2e^{-2t}(3\sin 5t + 10\cos 5t)$$

For the jerk, differentiate $a(t)$ using the product rule:

$$j(t) = a'(t) = \frac{d}{dt}[-2e^{-2t}(3\sin 5t + 10\cos 5t)]$$

$$= -2[-2e^{-2t}(3\sin 5t + 10\cos 5t) + e^{-2t}(15\cos 5t - 50\sin 5t)]$$

$$= -2e^{-2t}[-6\sin 5t - 20\cos 5t + 15\cos 5t - 50\sin 5t]$$

$$= -2e^{-2t}(-56\sin 5t - 5\cos 5t) = 2e^{-2t}(56\sin 5t + 5\cos 5t)$$

At $t = 0$:

$$j(0) = 2e^{0}(56 \cdot 0 + 5 \cdot 1) = 2 \times 5 = 10 \text{ m/s}^3$$

■

Exercises 6.2

6.2.1 A rocket's altitude is $h(t) = 80t^2 - 8/3\,t^3$ (m, $0 \le t \le 20$ s). Find: (a) velocity at $t = 4$ s; (b) acceleration at $t = 6$ s; (c) when velocity is maximum; (d) the jerk function and its value.

6.2.2 A piston follows $x(t) = 0.08\cos(40\pi t)$ metres. Find: (a) maximum velocity; (b) maximum acceleration; (c) jerk at $t = 1/80$ s.

6.2.3 Show that for $s(t) = A\sin(\omega t + \varphi)$, the acceleration satisfies $a = -\omega^2 s$. Explain the physical significance of this proportionality.

6.2.4 A particle's acceleration is $a(t) = 6t - 18$ m/s^2. If $v(0) = 4$ m/s and $s(0) = 0$, find $s(t)$ and the jerk at any time t.

6.2.5 A bungee cord gives a jumper the position $s(t) = 50 - 25e^{-0.5t}\cos(2t)$ m. Find the velocity and acceleration at $t = 0$ and interpret physically.

6.3 Concavity and Inflection Points

6.3.1 Definition of Concavity

The second derivative provides precise quantitative information about the shape — specifically the curvature — of a function's graph. This geometric property is called concavity.

Definition: A function f is said to be:

150. **Concave up** on an interval I if $f''(x) > 0$ for all $x \in I$.

151. **Concave down** on an interval I if $f''(x) < 0$ for all $x \in I$.

Geometric interpretation:

Concave up: The graph lies above all its tangent lines. The slope $f'(x)$ is increasing. Visually, the curve looks like the inside of a bowl (∪).

Concave down: The graph lies below all its tangent lines. The slope $f'(x)$ is decreasing. Visually, the curve looks like an upside-down bowl (∩).

6.3.2 Proof that $f'' > 0$ Implies Concave Up

Theorem: If $f''(x) > 0$ on I, then the graph of f lies above every tangent line on I.

Proof: Let $a \in I$. The tangent line at $\big(a, f(a)\big)$ is $L(x) = f(a) + f'(a)(x - a)$.

We want to show $f(x) > L(x)$ for $x \neq a$. By Taylor's theorem with Lagrange remainder:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c)}{2}(x - a)^2$$

for some c strictly between a and x. Therefore:

$$f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2$$

Since $f''(c) > 0$ (by assumption) and $(x - a)^2 > 0$ for $x \neq a$, we conclude $f(x) - L(x) > 0$, i.e. $f(x) > L(x)$ for all $x \neq a$. ▫

6.3.3 Inflection Points — Definition and Method

Definition: A point $\big(c, f(c)\big)$ is an *inflection point* of f if (i) f is continuous at c, and (ii) the concavity of f changes sign at c — that is, f'' changes from positive to negative or vice versa at $x = c$.

Necessary condition: If $\big(c, f(c)\big)$ is an inflection point and $f''(c)$ exists, then $f''(c) = 0$.

IMPORTANT CAUTION: $f''(c) = 0$ is necessary but **NOT** sufficient. We must verify the sign change. For example: $f(x) = x^4$ has $f''(0) = 0$ but does **NOT** have an inflection point at $x = 0$ because $f''(x) = 12x^2 \geq 0$ on both sides of 0 — no sign change occurs.

Procedure for finding inflection points:

Step 1: Compute $f''(x)$.

Step 2: Solve $f''(x) = 0$ and identify points where f'' is undefined.

Step 3: Divide the real line into intervals at these candidate points.

Step 4: Test the sign of f'' in each interval.

Step 5: An inflection point exists at $x = c$ if and only if f'' changes sign at c.

6.3.4 The Second Derivative Test for Local Extrema

Theorem (Second Derivative Test): Suppose $f'(c) = 0$ and $f''(c)$ exists.

152. If $f''(c) > 0$, then f has a **local minimum** at c.

153. If $f''(c) < 0$, then f has a **local maximum** at c.

154. If $f''(c) = 0$, the test is **inconclusive** — use the first derivative test.

Proof of the minimum case: Suppose $f'(c) = 0$ and $f''(c) > 0$. Since $f''(c) > 0$, by definition of the derivative:

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h} > 0$$

This means for sufficiently small $|h| > 0$: $f'(c+h)/h > 0$. For $h > 0$ this gives $f'(c+h) > 0$ (increasing to the right of c), and for $h < 0$ this gives $f'(c+h) < 0$ (decreasing to the left of c). By the First Derivative Test, f has a local minimum at c. ▫

6.3.5 Higher-Order Derivative Test (Inconclusive Cases)

When $f'(c) = 0$ and $f''(c) = 0$ simultaneously, we examine higher derivatives:

Theorem: Suppose $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. Then:

155. If n is **even** and $f^{(n)}(c) > 0$: **local minimum** at c.

156. If n is **even** and $f^{(n)}(c) < 0$: **local maximum** at c.

157. If n is **odd**: **neither** (a horizontal inflection point).

Example: $f(x) = x^5$. We have $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0$, but $f^{(5)}(0) = 120 \neq 0$ and $n = 5$ is odd. Therefore $x = 0$ is a horizontal inflection point — consistent with the graph of x^5.

Worked Examples — Section 6.3

Example 6.3.1 — Full Curve Analysis

Analyse $f(x) = x^4 - 8x^3 + 18x^2 - 10$ for concavity, inflection points, and extrema.

Solution:

$$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$$

$$f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-1)(x-3)$$

Critical points ($f'(x) = 0$): $x = 0$ and $x = 3$.

$f''(0) = 12(-1)(-3) = 36 > 0 \rightarrow$ **local minimum** at $x = 0$. $f(0) = -10$.

$f''(3) = 0 \rightarrow$ inconclusive; use higher-order test.

$f'''(x) = 24x - 48$; $f'''(3) = 24 \neq 0$, so $n = 3$ (odd) $\rightarrow x = 3$ is a **horizontal inflection point**.

Inflection points ($f''(x) = 0$): $x = 1$ and $x = 3$.

Sign of $f'' = 12(x-1)(x-3)$:

158. $x < 1$: $(+)(-)(-) = (+) > 0 \rightarrow$ concave up

159. $1 < x < 3$: $(+)(+)(-) = (-) < 0 \rightarrow$ concave down

160. $x > 3$: $(+)(+)(+) = (+) > 0 \rightarrow$ concave up

f'' changes sign at both $x = 1$ and $x = 3 \rightarrow$ both are inflection points.

$f(1) = 1 - 8 + 18 - 10 = 1$; point $(1, 1)$ is an inflection point.

$f(3) = 81 - 216 + 162 - 10 = 17$; point $(3, 17)$ is the horizontal inflection point.

Summary: Local minimum at $(0, -10)$. Inflection points at $(1, 1)$ and $(3, 17)$.

■

Example 6.3.2 — Inflection Points of a Transcendental Function

Find the concavity and inflection points of $f(x) = xe^{-x}$.

Solution:

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$$

$$f''(x) = -e^{-x}(1 - x) + e^{-x}(-1) = e^{-x}(-1 + x - 1) = e^{-x}(x - 2)$$

Since $e^{-x} > 0$ always, the sign of f'' is determined entirely by $(x - 2)$:

161. $x < 2$: $f''(x) < 0 \rightarrow$ concave down

162. $x > 2$: $f''(x) > 0 \rightarrow$ concave up

163. $x = 2$: $f''(2) = 0$ and f'' changes sign $\rightarrow$ inflection point

$$f(2) = 2e^{-2} \approx 0.2707$$

Inflection point at $(2,\ 2e^{-2}) \approx (2,\ 0.271)$**.**

■

Example 6.3.3 — Optimisation with Second Derivative Test

A manufacturing company's weekly profit (thousands of dollars) is:

$$P(x) = -2x^3 + 15x^2 - 24x + 5 \qquad (x = \text{units produced, hundreds})$$

Find the production level that maximises profit and verify using the second derivative test.

Solution:

$$P'(x) = -6x^2 + 30x - 24 = -6(x^2 - 5x + 4) = -6(x - 1)(x - 4)$$

$P'(x) = 0 \Longrightarrow x = 1$ and $x = 4$.

$$P''(x) = -12x + 30$$

$P''(1) = 18 > 0 \rightarrow$ **local minimum** at $x = 1$. $P(1) = -2 + 15 - 24 + 5 = -6$.

$P''(4) = -18 < 0 \rightarrow$ **local maximum** at $x = 4$. $P(4) = -128 + 240 - 96 + 5 = 21$.

Inflection point ($P''(x) = 0$): $x = 2.5$.

$$P(2.5) = -2(15.625) + 15(6.25) - 24(2.5) + 5 = -31.25 + 93.75 - 60 + 5 = 7.5$$

Maximum weekly profit of \$21,000 is achieved at $x = 4$ **(400 units per week).** The inflection point at $(2.5,\ 7.5)$ marks where the rate of profit growth switches from increasing to decreasing — the "diminishing returns" threshold.

■

Exercises 6.3

6.3.1 Find intervals of concavity and inflection points for: (a) $f(x) = x^3 - 3x^2 + 4$; (b) $f(x) = \sin x$ on $[0, 2\pi]$; (c) $f(x) = x^2 e^{-x}$.

6.3.2 Apply the second derivative test to classify all critical points of $f(x) = 3x^4 - 16x^3 + 24x^2 - 5$.

6.3.3 Show that $f(x) = x^4$ has a local minimum at $x = 0$ even though $f''(0) = 0$. Use the higher-order derivative test and verify graphically.

6.3.4 For $f(x) = x^5 - 5x$, find all local extrema and inflection points.

6.3.5 A beam's bending moment is $M(x) = 15x - x^2 - 20$ (kN·m, x in m, $0 \le x \le 10$). Find the location of maximum bending moment and classify it using the second derivative test.

6.3.6 Prove that a polynomial of odd degree with positive leading coefficient must have at least one inflection point.

6.4 The nth Derivative and the Leibniz Rule

6.4.1 The Challenge of nth Derivatives of Products

For the nth derivative of a product of two functions, repeated application of the ordinary product rule quickly becomes cumbersome. For example, finding $(x^4 \sin x)^{(8)}$ by successive differentiation would require eight applications of the product rule. The Leibniz Rule provides an elegant closed-form formula that solves this problem in one step, using the same structure as the Binomial Theorem.

6.4.2 Statement of the Leibniz Rule

Theorem (General Leibniz Rule): If f and g are n times differentiable functions on an interval I, then their product fg is also n times differentiable, and:

$$\boxed{(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ are the binomial coefficients. Written in expanded form:

$$(fg)^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2!}f^{(n-2)}g'' + \cdots + nf\,g^{(n-1)} + fg^{(n)}$$

6.4.3 Derivation by Mathematical Induction

Proof:

Base case ($n = 1$): The formula gives $\binom{1}{0}f^{(0)}g^{(1)} + \binom{1}{1}f^{(1)}g^{(0)} = fg' + f'g$, which is exactly the standard product rule. ✓

Inductive step: Assume the formula holds for $n = k$:

$$(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$$

Differentiating both sides with respect to x:

$$(fg)^{(k+1)} = \sum_{j=0}^{k} \binom{k}{j} \left[f^{(j+1)} g^{(k-j)} + f^{(j)} g^{(k-j+1)} \right]$$

Splitting into two sums and re-indexing the first (let $m = j + 1$):

$$= \sum_{m=1}^{k+1} \binom{k}{m-1} f^{(m)} g^{(k+1-m)} + \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k+1-j)}$$

Collecting terms for each m from 0 to $k + 1$ and applying Pascal's identity $\binom{k}{m-1} + \binom{k}{m} = \binom{k+1}{m}$, we obtain:

$$(fg)^{(k+1)} = \sum_{m=0}^{k+1} \binom{k+1}{m} f^{(m)} g^{(k+1-m)}$$

This is precisely the Leibniz formula for $n = k + 1$. ▫

The structural identity with the Binomial Theorem is not coincidental — both arise from the same combinatorial counting argument, with differentiation replacing multiplication.

6.4.4 Strategy for Applying the Leibniz Rule

Assign f to the function whose derivatives eventually become zero or simplify significantly (typically a polynomial), and g to the function whose derivatives are easy to compute in general (typically an exponential or trigonometric function). This minimises the number of nonzero terms in the sum.

Worked Examples — Section 6.4

Example 6.4.1 — Leibniz Rule: Polynomial × Exponential

Find the nth derivative of $h(x) = x^3 e^x$.

Solution: Let $f(x) = x^3$ (polynomial) and $g(x) = e^x$.

Derivatives of f: $f' = 3x^2$, $f'' = 6x$, $f''' = 6$, $f^{(k)} = 0$ for $k \geq 4$.

Derivatives of g: $g^{(k)}(x) = e^x$ for all k.

By the Leibniz rule (only $k = 0,1,2,3$ contribute since $f^{(k)} = 0$ for $k \geq 4$):

$$h^{(n)}(x) = \binom{n}{0} x^3 e^x + \binom{n}{1} 3x^2 e^x + \binom{n}{2} 6x\, e^x + \binom{n}{3} 6\, e^x$$

$$= e^x[x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2)] \qquad (n \geq 3)$$

Verification $(n = 2)$: $h''(x) = e^x(x^3 + 6x^2 + 6x)$.

Direct check: $h'(x) = e^x(x^3 + 3x^2)$, $h''(x) = e^x(x^3 + 3x^2) + e^x(3x^2 + 6x) = e^x(x^3 + 6x^2 + 6x)$. ✓

■

Example 6.4.2 — Leibniz Rule: Polynomial × Sine

Find $h^{(5)}(x)$ for $h(x) = x^2 \sin x$.

Solution: Let $f(x) = x^2$ and $g(x) = \sin x$.

Derivatives of f: $f = x^2$, $f' = 2x$, $f'' = 2$, $f^{(k)} = 0$ for $k \geq 3$.

Derivatives of g: $g^{(k)}(x) = \sin(x + k\pi/2)$.

$$g^{(5)}(x) = \sin(x + 5\pi/2) = \sin(x + 2\pi + \pi/2) = \cos x$$

$$g^{(4)}(x) = \sin(x + 4\pi/2) = \sin(x + 2\pi) = \sin x$$

$$g^{(3)}(x) = \sin(x + 3\pi/2) = -\cos x$$

Applying Leibniz with $n = 5$:

$$h^{(5)}(x) = \binom{5}{0} x^2 \cos x + \binom{5}{1} 2x \sin x + \binom{5}{2} 2(-\cos x)$$

$$= x^2 \cos x + 10x \sin x - 20 \cos x$$

$$= \boxed{(x^2 - 20)\cos x + 10x \sin x}$$

■

Example 6.4.3 — General Formula: $e^x \cos x$

Derive the general nth derivative formula for $h(x) = e^x \cos x$.

Solution — using polar form of complex numbers:

Write $e^x \cos x = \mathrm{Re}\left(e^x \cdot e^{ix}\right) = \mathrm{Re}\left(e^{(1+i)x}\right)$.

The nth derivative of $e^{(1+i)x}$ is $(1 + i)^n e^{(1+i)x}$.

In polar form: $(1 + i) = \sqrt{2}\, e^{i\pi/4}$, so $(1 + i)^n = 2^{n/2} e^{in\pi/4}$.

Therefore:

$$h^{(n)}(x) = \mathrm{Re}\left[2^{n/2} e^{in\pi/4} \cdot e^{(1+i)x}\right] = \mathrm{Re}\left[2^{n/2} e^x\, e^{i(x+n\pi/4)}\right]$$

$$= \boxed{2^{n/2}\, e^{x} \cos\left(x + \frac{n\pi}{4}\right)}$$

Verification ($n = 1$): $\sqrt{2}\, e^{x}\cos(x + \pi/4) = \sqrt{2}\, e^{x}\left[\frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}}\right] = e^{x}(\cos x - \sin x)$.

Direct: $h'(x) = e^{x}\cos x - e^{x}\sin x = e^{x}(\cos x - \sin x)$. ✓

Verification ($n = 2$): $2e^{x}\cos(x + \pi/2) = 2e^{x}(-\sin x) = -2e^{x}\sin x$.

Direct: $h''(x) = (e^{x}\cos x - e^{x}\sin x)' = e^{x}\cos x - e^{x}\sin x - e^{x}\sin x - e^{x}\cos x = -2e^{x}\sin x$. ✓

■

Example 6.4.4 — Leibniz Applied to $x^4 e^{2x}$

Find $h^{(6)}(x)$ for $h(x) = x^4 e^{2x}$.

Solution: $f = x^4$, $g = e^{2x}$, $g^{(k)} = 2^k e^{2x}$.

Derivatives of f: $f' = 4x^3$, $f'' = 12x^2$, $f''' = 24x$, $f^{(4)} = 24$, $f^{(k)} = 0$ for $k \geq 5$.

With $n = 6$ (only $k = 0,1,2,3,4$ contribute):

$$h^{(6)} = \binom{6}{0} x^4 \cdot 64e^{2x} + \binom{6}{1} 4x^3 \cdot 32e^{2x} + \binom{6}{2} 12x^2 \cdot 16e^{2x} + \binom{6}{3} 24x \cdot 8e^{2x} + \binom{6}{4} 24 \cdot 4e^{2x}$$

$$= e^{2x}[64x^4 + 768x^3 + 2880x^2 + 3840x + 1440]$$

$$= 32e^{2x}[2x^4 + 24x^3 + 90x^2 + 120x + 45]$$

■

Exercises 6.4

6.4.1 Use the Leibniz rule to find $h^{(n)}(x)$ for: (a) $h(x) = x^2 e^{x}$; (b) $h(x) = x^3 e^{-x}$; (c) $h(x) = x\sin x$.

6.4.2 Find the 6th derivative of $h(x) = x^3\cos(2x)$ using the Leibniz rule.

6.4.3 Derive the general formula $(e^{x}\sin x)^{(n)} = 2^{n/2} e^{x}\sin(x + n\pi/4)$.

6.4.4 Verify the Leibniz rule for $n = 3$ applied to $h(x) = x^2 \ln x$ by: (a) using the Leibniz formula directly; (b) direct successive differentiation.

6.4.5 Find $h^{(8)}(0)$ for $h(x) = x^4\cos x$. [*Hint:* use Leibniz; evaluate at $x = 0$.]

6.4.6 *(Challenge)* Use the Leibniz rule to prove that if $p(x)$ is a degree-m polynomial and $n > m$, then $\big(p(x)\, g(x)\big)^{(n)} = \sum_{k=0}^{m} \binom{n}{k} p^{(k)}(x)\, g^{(n-k)}(x)$.

6.5 Engineering Applications

6.5.1 Beam Deflection — The Euler-Bernoulli Theory

Background and Fundamental Assumptions

The Euler-Bernoulli beam theory is the classical framework for analysing the transverse bending of slender structural members and represents one of the most important applications of fourth-order differential equations in civil and mechanical engineering. The theory was developed by Daniel Bernoulli and Leonhard Euler in the 18th century and remains the foundation of structural analysis to this day.

The theory rests on five fundamental assumptions:

164. The beam is initially straight, prismatic (uniform cross-section), and linearly elastic.

165. Deflections are small compared to the beam's length (small-deflection, or linear, theory).

166. The material is homogeneous and isotropic, obeying Hooke's Law.

167. Plane cross-sections remain plane and perpendicular to the neutral axis after bending (Bernoulli's hypothesis).

168. Transverse loads act in the plane of symmetry of the cross-section.

The Moment-Curvature Relationship

For a beam deflecting in the xy-plane, the exact curvature at any point is:

$$\kappa = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

Under the small-deflection assumption, $|dy/dx| \ll 1$, so $(dy/dx)^2 \approx 0$ and the curvature simplifies to:

$$\kappa \approx \frac{d^2y}{dx^2} = y''$$

The bending moment M is related to curvature by the fundamental moment-curvature law:

$$M(x) = EI\, y''(x)$$

where E is the elastic (Young's) modulus [Pa = N/m^2] and I is the second moment of area of the cross-section about its neutral axis [m^4]. The product EI is called the **flexural rigidity** of the beam.

Derivation of the Complete Governing Equation

From statics of an infinitesimal beam element, two equilibrium relationships hold:

$$\frac{dM}{dx} = V(x) \qquad \text{(shear force equals rate of change of moment)}$$

$$\frac{dV}{dx} = -w(x) \qquad \text{(distributed load equals rate of change of shear)}$$

Differentiating $M = EI\,y''$ with respect to x:

$$EI\,y''' = V(x)$$

Differentiating again:

$$EI\,y^{(4)} = -w(x)$$

The Euler-Bernoulli Beam Equation:

$$\boxed{EI\,\frac{d^4y}{dx^4} = w(x)}$$

(using the convention that downward distributed load w is positive and upward deflection y is positive; some textbooks reverse this sign convention).

This 4th-order ODE requires four boundary conditions. The complete hierarchy of derivatives and their physical meaning is:

Derivative	Quantity	Symbol	Physical Meaning
$y(x)$	Deflection	[m]	Transverse displacement of beam axis
$y'(x)$	Slope	[rad]	Angle of inclination of elastic curve
$EI \cdot y''(x)$	Bending Moment	$M(x)$ [N·m]	Internal moment — causes bending stress
$EI \cdot y'''(x)$	Shear Force	$V(x)$ [N]	Internal transverse force
$EI \cdot y^{(4)}(x)$	Distributed Load	$w(x)$ [N/m]	External load per unit length

Boundary Conditions for Common Support Types

Support Type	Boundary Conditions at $x = a$
Simply supported (pin/roller)	$y(a) = 0$ (no deflection), $y''(a) = 0$ (no moment)
Fixed / Clamped (cantilever wall)	$y(a) = 0$ (no deflection), $y'(a) = 0$ (no slope)
Free end	$y''(a) = 0$ (no moment), $y'''(a) = 0$ (no shear)
Free end with point load P	$y''(a) = 0$, $EI \cdot y'''(a) = \mp P$ (shear = applied load)

Worked Examples — Section 6.5.1

Example 6.5.1 — Simply Supported Beam with Uniform Load

Derive the deflection curve for a simply supported beam of length L carrying a uniformly distributed load of intensity w_0 (N/m). Find the maximum deflection.

Setup: $EI\,y^{(4)} = w_0$ (constant, downward load positive, upward y positive).

BCs: $y(0) = 0,\ y''(0) = 0,\ y(L) = 0,\ y''(L) = 0$.

Step 1 — Integrate four times:

$$EI\,y''' = w_0 x + C_1$$

$$EI\,y'' = \frac{w_0}{2}x^2 + C_1 x + C_2$$

$$EI\,y' = \frac{w_0}{6}x^3 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$EI\,y = \frac{w_0}{24}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

Step 2 — Apply boundary conditions:

BC1: $y(0) = 0 \Longrightarrow C_4 = 0$.

BC2: $y''(0) = 0 \Longrightarrow C_2 = 0$.

BC3: $y''(L) = 0 \Longrightarrow \frac{w_0}{2}L^2 + C_1 L = 0 \Longrightarrow C_1 = -\frac{w_0 L}{2}$.

BC4: $y(L) = 0$:

$$\frac{w_0}{24}L^4 - \frac{w_0 L}{2}\cdot\frac{L^3}{6} + C_3 L = 0 \Longrightarrow \frac{w_0 L^4}{24} - \frac{w_0 L^4}{12} + C_3 L = 0 \Longrightarrow C_3 = \frac{w_0 L^3}{24}$$

Step 3 — Deflection equation:

$$EI\,y = \frac{w_0}{24}x^4 - \frac{w_0 L}{12}x^3 + \frac{w_0 L^3}{24}x = \frac{w_0 x}{24}(x^3 - 2Lx^2 + L^3)$$

$$\boxed{y(x) = \frac{w_0 x(x^3 - 2Lx^2 + L^3)}{24EI}}$$

Step 4 — Maximum deflection at $x = L/2$ (by symmetry):

$$y\left(\frac{L}{2}\right) = \frac{w_0(L/2)}{24EI}\left[\frac{L^3}{8} - \frac{L^3}{2} + L^3\right] = \frac{w_0 L}{48EI}\cdot\frac{5L^3}{8}$$

$$\boxed{|y_{\max}| = \frac{5w_0 L^4}{384EI}} \qquad \leftarrow \text{standard result}$$

■

Example 6.5.2 — Cantilever Beam: Point Load at Free End

A cantilever of length L is clamped at $x = 0$ and free at $x = L$. A downward point load P is applied at the free end. Find the deflection curve.

Setup: No distributed load: $EI\, y^{(4)} = 0$.

BCs: $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$ (no moment at free end), $EI\, y'''(L) = P$ (shear = P).

Integrating $EI\, y^{(4)} = 0$ four times:

$$EI\, y''' = C_1, \quad EI\, y'' = C_1 x + C_2, \quad EI\, y' = \frac{C_1}{2}x^2 + C_2 x + C_3, \quad EI\, y = \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

Applying BCs:

$y(0) = 0 \Longrightarrow C_4 = 0; \quad y'(0) = 0 \Longrightarrow C_3 = 0.$

$EI\, y'''(L) = P \Longrightarrow C_1 = P; \quad y''(L) = 0 \Longrightarrow PL + C_2 = 0 \Longrightarrow C_2 = -PL.$

Deflection equation:

$$EI\, y = \frac{P}{6}x^3 - \frac{PL}{2}x^2 = \frac{Px^2}{6}(x - 3L)$$

$$\boxed{y(x) = \frac{Px^2(x - 3L)}{6EI}} \qquad \text{(downward deflection for } 0 \le x \le L\text{)}$$

Maximum deflection at $x = L$:

$$y(L) = \frac{PL^2(L - 3L)}{6EI} = \frac{-2PL^3}{6EI} \Longrightarrow \boxed{|y_{\max}| = \frac{PL^3}{3EI}} \qquad \leftarrow \text{standard cantilever result}$$

Slope at free end:

$$EI\, y'(L) = P \cdot \frac{L^2}{2} - PL \cdot L = -\frac{PL^2}{2} \Longrightarrow \theta_{\max} = \frac{PL^2}{2EI} \qquad \text{(rotation at free end)}$$

■

Example 6.5.3 — Fixed-Fixed Beam: Uniform Load

A beam is clamped at both ends ($x = 0$ and $x = L$) with uniform load w_0. Find $y(x)$.

BCs: $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y'(L) = 0$.

Integrating $EI\, y^{(4)} = w_0$:

$$EI\, y = \frac{w_0}{24}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

BC1: $y(0) = 0 \Longrightarrow C_4 = 0$; BC2: $y'(0) = 0 \Longrightarrow C_3 = 0$.

From $y'(L) = 0$: $\frac{w_0}{6}L^3 + \frac{C_1}{2}L^2 + C_2 L = 0 \quad \cdots$ (i)

From $y(L) = 0$: $\frac{w_0}{24}L^4 + \frac{C_1}{6}L^3 + \frac{C_2}{2}L^2 = 0 \quad \cdots$ (ii)

Solving equations (i) and (ii): $C_1 = -\frac{w_0 L}{2}$, $C_2 = \frac{w_0 L^2}{12}$.

$$\boxed{y(x) = \frac{w_0 x^2 (x - L)^2}{24EI}}$$

Maximum deflection at $x = L/2$:

$$y\left(\frac{L}{2}\right) = \frac{w_0 (L/2)^2 (L/2 - L)^2}{24EI} = \frac{w_0 \cdot (L^2/4) \cdot (L^2/4)}{24EI} = \frac{w_0 L^4}{384EI}$$

Comparison with simply supported beam:

$$\text{Simply supported: } |y_{\max}| = \frac{5 w_0 L^4}{384EI} \qquad \text{Fixed-fixed: } |y_{\max}| = \frac{w_0 L^4}{384EI}$$

The fixed-fixed beam is **5 times stiffer** — the clamped ends provide enormous additional restraint.

■

Example 6.5.4 — Numerical Beam Design

A simply supported steel floor beam spans $L = 8$ m and carries $w_0 = 20$ kN/m. $E = 200$ GPa. The allowable midspan deflection is $L/360$. Find the minimum required second moment of area I.

Solution:

Maximum deflection formula: $|y_{\max}| = \frac{5 w_0 L^4}{384EI}$

Allowable deflection: $\Delta_{\text{allow}} = \frac{L}{360} = \frac{8000}{360} = 22.22 \text{ mm} = 0.02222 \text{ m}$

Setting $|y_{\max}| \le \Delta_{\text{allow}}$:

$$\frac{5 \times 20{,}000 \times 8^4}{384 \times 200 \times 10^9 \times I} \le 0.02222$$

Numerator: $5 \times 20{,}000 \times 4096 = 409{,}600{,}000 \text{ N·m}^3$

Denominator factor: $384 \times 200 \times 10^9 = 7.68 \times 10^{13}$

$$I \ge \frac{409{,}600{,}000}{7.68 \times 10^{13} \times 0.02222} = \frac{409{,}600{,}000}{1{,}706{,}534{,}400} \ge 2.40 \times 10^{-4} \text{ m}^4 = 240 \times 10^6 \text{ mm}^4$$

Minimum required $I = 2.40 \times 10^{-4}$ **m**^{4} (a standard UB 406\$178\$74 section with $I \approx 274 \times 10^6$ mm^4 would suffice).

■

Exercises 6.5.1 — Beam Deflection

6.5.1 A simply supported beam, $L = 5$ m, $EI = 6 \times 10^6$ N·m^2, carries $w_0 = 18$ kN/m. Find: (a) deflection equation; (b) midspan deflection; (c) slope at both supports.

6.5.2 A cantilever beam, $L = 3$ m, carries $w_0 = 12$ kN/m (uniform, downward). $EI = 4 \times 10^6$ N·m^2. Derive the deflection equation and find: (a) maximum deflection at free end; (b) slope at the free end.

6.5.3 A simply supported beam carries a central point load $P = 40$ kN at $x = L/2$. $L = 6$ m, $EI = 10 \times 10^6$ N·m^2. Find the maximum deflection. [Answer: $PL^3/(48EI)$]

6.5.4 Show that the bending moment in a simply supported beam with uniform load w_0 is $M(x) = EI \cdot y''(x) = w_0 x(L - x)/2$. Verify it is maximum at $x = L/2$.

6.5.5 A propped cantilever is fixed at $x = 0$ and simply supported at $x = L$ with uniform w_0. State the four boundary conditions and set up (but do not fully solve) the system.

6.5.6 *(Design)* A timber beam ($E = 12$ GPa) spans $L = 5$ m, simply supported, carrying 15 kN/m. Allowable deflection $= L/300$. Find minimum I required.

6.5.2 Vibrations and Oscillations

6.6.2.1 Fundamentals of Oscillatory Systems

Mechanical vibrations are encountered in virtually every branch of engineering — from the seismic response of buildings, to the dynamic behaviour of rotating machinery, to the flutter of aircraft wings. The mathematical description of vibrations relies fundamentally on second and higher-order derivatives. The general governing equation of a single-degree-of-freedom (SDOF) mass-spring-damper system encapsulates the interplay of inertia (second derivative), damping (first derivative), and stiffness (zeroth derivative).

6.6.2.2 Undamped Free Vibration — Derivation of the Equation of Motion

Consider a mass m (kg) on a frictionless surface attached to a spring of stiffness k (N/m). Let x denote displacement from the equilibrium position. The spring exerts a restoring force $F_s = -kx$. Applying Newton's second law:

$$\sum F = ma \Longrightarrow -kx = m\ddot{x}$$

Rearranging to standard form:

$$m\ddot{x} + kx = 0$$

Dividing by m and defining the natural frequency $\omega_n = \sqrt{k/m}$:

$$\boxed{\ddot{x} + \omega_n^2 x = 0} \qquad \text{(Undamped SHM equation)}$$

The general solution is:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t) = C\cos(\omega_n t - \varphi)$$

where A, B (or equivalently C and φ) are determined by initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$A = x_0, \qquad B = \frac{v_0}{\omega_n}, \qquad C = \sqrt{x_0^2 + \frac{v_0^2}{\omega_n^2}}$$

Key relationships:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} \qquad T = \frac{2\pi}{\omega_n} \text{ (natural period, s)} \qquad f_n = \frac{\omega_n}{2\pi} \text{ (natural frequency, Hz)}$$

The velocity and acceleration are:

$$\dot{x}(t) = -A\omega_n\sin(\omega_n t) + B\omega_n\cos(\omega_n t)$$

$$\ddot{x}(t) = -A\omega_n^2\cos(\omega_n t) - B\omega_n^2\sin(\omega_n t) = -\omega_n^2\, x(t)$$

The relation $\ddot{x} = -\omega_n^2 x$ verifies that the solution satisfies the equation of motion and shows that the restoring acceleration is always proportional to — and opposite in direction to — the displacement.

6.6.2.3 Damped Free Vibration

In practice, all vibrating systems dissipate energy through friction, internal damping in materials, or fluid resistance. For a linear viscous damper (dashpot) with damping coefficient c (N·s/m), the damping force is $F_d = -c\dot{x}$. Newton's second law gives:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Dividing by m:

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

where two fundamental parameters are defined:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{(undamped natural frequency, rad/s)}$$

$$\zeta = \frac{c}{2\sqrt{km}} \qquad \text{(damping ratio, dimensionless)}$$

The critical damping coefficient is $c_c = 2\sqrt{km} = 2m\omega_n$, and $\zeta = c/c_c$.

The characteristic equation (assuming $x = e^{st}$):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \Longrightarrow s = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Three physically distinct cases arise:

Case 1: Underdamped ($\zeta < 1$)

Complex conjugate roots: $s = -\zeta\omega_n \pm i\omega_d$, where the damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

$$x(t) = e^{-\zeta\omega_n t}[A\cos(\omega_d t) + B\sin(\omega_d t)] = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \varphi)$$

The motion is oscillatory with exponentially decaying amplitude. The envelope of the oscillation is $Ce^{-\zeta\omega_n t}$. This is the most common situation in practice (most structures and machines are underdamped).

Case 2: Critically Damped ($\zeta = 1$)

Repeated real root: $s = -\omega_n$ (double root).

$$x(t) = (A + Bt)\, e^{-\omega_n t}$$

The system returns to equilibrium as rapidly as possible without oscillating. Used in precision instruments, electrical meter movements, and hydraulic door closers.

Case 3: Overdamped ($\zeta > 1$)

Two distinct negative real roots: $s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$.

$$x(t) = Ae^{s_1 t} + Be^{s_2 t}$$

The system decays exponentially without oscillation, but more slowly than the critically damped case.

6.6.2.4 Forced Vibration with Harmonic Excitation

When an external harmonic force $F(t) = F_0\cos(\Omega t)$ is applied:

$$m\ddot{x} + c\dot{x} + kx = F_0\cos(\Omega t)$$

The complete solution has two parts: $x(t) = x_h(t) + x_p(t)$, where x_h is the transient (homogeneous) solution that decays with time, and x_p is the steady-state (particular) solution:

$$x_p(t) = X\cos(\Omega t - \varphi)$$

where the steady-state amplitude X and phase φ are:

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \varphi = \arctan\left[\frac{2\zeta r}{1-r^2}\right]$$

and $r = \Omega/\omega_n$ is the frequency ratio. The Dynamic Magnification Factor (DMF) is:

$$\text{DMF} = \frac{X}{F_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

At resonance ($r \to 1$): DMF $\to 1/(2\zeta)$, which approaches infinity as $\zeta \to 0$. This is why resonance is catastrophic in undamped systems. Even modest levels of damping ($\zeta = 0.05$) limit the amplitude to DMF $= 10$ at resonance.

6.5.2.5 Logarithmic Decrement

For underdamped free vibration, the damping ratio can be measured experimentally using successive amplitude peaks. If x_1 and x_2 are the amplitudes of two successive peaks (separated by one damped period $T_d = 2\pi/\omega_d$):

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

This is the **logarithmic decrement**. For small ζ, $\delta \approx 2\pi\zeta$. This formula is widely used in experimental vibration testing to extract damping ratios from measured decay curves.

Worked Examples — Section 6.5.2

Example 6.5.5 — Undamped Mass-Spring System

A machine component of mass $m = 8$ kg is mounted on a spring of stiffness $k = 3200$ N/m. Initial conditions: $x(0) = 0.05$ m, $\dot{x}(0) = 0.4$ m/s.

169. Find ω_n, period T, and natural frequency f_n. (b) Write the complete solution $x(t)$. (c) Find the amplitude C. (d) Find velocity and acceleration at $t = 0.1$ s. (e) Find maximum acceleration and the time it first occurs.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3200}{8}} = \sqrt{400} = 20 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0.3142 \text{ s} \qquad f_n = \frac{20}{2\pi} \approx 3.183 \text{ Hz}$$

170. $x(t) = A\cos(20t) + B\sin(20t)$

$x(0) = A = 0.05$ m; $\quad \dot{x}(0) = 20B = 0.4 \Longrightarrow B = 0.02$ m.

$$x(t) = 0.05\cos(20t) + 0.02\sin(20t) \text{ m}$$

171. $C = \sqrt{A^2 + B^2} = \sqrt{0.0025 + 0.0004} = \sqrt{0.0029} \approx 0.05385$ m ≈ 53.85 mm.

172. At $t = 0.1$ s:

$$\dot{x}(t) = -\sin(20t) + 0.4\cos(20t)$$

$$\dot{x}(0.1) = -\sin 2 + 0.4\cos 2 = -0.9093 + 0.4(-0.4161) = -1.076 \text{ m/s}$$

$$\ddot{x}(t) = -\omega_n^2\, x(t) = -400\, x(t)$$

$$x(0.1) = 0.05\cos 2 + 0.02\sin 2 = 0.05(-0.4161) + 0.02(0.9093) = -0.00262 \text{ m}$$

$$\ddot{x}(0.1) = -400(-0.00262) = 1.048 \text{ m/s}^2$$

173. Maximum acceleration: $|\ddot{x}|_{\max} = \omega_n^2 C = 400 \times 0.05385 = 21.54$ m/s^2.

First occurrence when $x(t) = -C$ (most negative), i.e. $\cos(20t - \varphi) = -1$, where $\varphi = \arctan(B/A) = \arctan(0.4) \approx 0.3805$ rad:

$$20t - 0.3805 = \pi \Longrightarrow t = \frac{\pi + 0.3805}{20} \approx 0.1760 \text{ s}$$

■

Example 6.5.6 — Damped Vibration: Vehicle Suspension

A vehicle suspension has: $m = 350$ kg, $k = 35{,}000$ N/m, $c = 3500$ N·s/m.

Initial: $x(0) = 0.06$ m, $\dot{x}(0) = 0$.

174. Find ω_n, ζ, ω_d. Classify the system. (b) Write the complete response $x(t)$. (c) Find the first peak amplitude and time to first peak. (d) Calculate logarithmic decrement and ratio of successive peak amplitudes.

Solution:

$$\omega_n = \sqrt{\frac{35{,}000}{350}} = \sqrt{100} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{3500}{2 \times 350 \times 10} = 0.5 \qquad \omega_d = 10\sqrt{1 - 0.25} = 10\sqrt{0.75} = 8.660 \text{ rad/s}$$

$\zeta = 0.5 < 1 \rightarrow$ **UNDERDAMPED**.

175. $x(t) = e^{-5t}[A\cos(8.660t) + B\sin(8.660t)]$

$x(0) = A = 0.06$ m.

$\dot{x}(0) = -5A + B\omega_d = 0 \Longrightarrow B = \frac{5 \times 0.06}{8.660} = 0.03464$ m.

$$x(t) = e^{-5t}[0.06\cos(8.660t) + 0.03464\sin(8.660t)] \text{ m}$$

176. First peak when $\dot{x} = 0$ (after $t = 0$):

$$\tan(\omega_d t) = \frac{\omega_d}{\zeta\omega_n} = \frac{8.660}{5} = 1.7321 \Longrightarrow \omega_d t = \frac{\pi}{3} \Longrightarrow t_{\text{peak}} = \frac{\pi/3}{8.660} \approx 0.1209 \text{ s}$$

$$x(t_{\text{peak}}) = e^{-5\times0.1209}[0.06\cos(\pi/3) + 0.03464\sin(\pi/3)] = e^{-0.6046}(0.03 + 0.03000)$$
$$= 0.5464 \times 0.06000 = 0.03278 \text{ m}$$

177. $T_d = 2\pi/\omega_d = 0.7255$ s.

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.5)}{\sqrt{0.75}} = \frac{\pi}{0.8660} = 3.628$$

$$\frac{x_2}{x_1} = e^{-\delta} = e^{-3.628} = 0.02664$$

Each peak is only 2.66% of the previous — the suspension damps very rapidly.

■

Example 6.5.7 — Resonance in Forced Vibration

A machine ($m = 120$ kg) is mounted on a spring-damper isolator: $k = 120{,}000$ N/m, $c = 960$ N·s/m. Unbalance force: $F(t) = 800\cos(\Omega t)$ N, with $\Omega = 31.4$ rad/s (≈ 300 RPM).

178. Find ω_n and ζ. (b) Find the frequency ratio r. (c) Calculate steady-state amplitude X. (d) Determine phase angle φ. (e) Compare X with static deflection and comment.

Solution:

$$\omega_n = \sqrt{\frac{120{,}000}{120}} = \sqrt{1000} = 31.62 \text{ rad/s} \qquad \zeta = \frac{960}{2 \times 120 \times 31.62} = 0.1265$$

179. $r = \Omega/\omega_n = 31.4/31.62 = 0.9930$ (very close to resonance).

$$1 - r^2 = 1 - 0.9860 = 0.0140 \qquad 2\zeta r = 2(0.1265)(0.9930) = 0.2512$$

$$\text{DMF} = \frac{1}{\sqrt{(0.0140)^2 + (0.2512)^2}} = \frac{1}{\sqrt{0.06330}} = 3.975$$

Static deflection: $\delta_{st} = F_0/k = 800/120{,}000 = 0.006667$ m.

$$X = \text{DMF} \times \delta_{st} = 3.975 \times 0.006667 = 0.02650 \text{ m} = 26.50 \text{ mm}$$

180. $\varphi = \arctan\left[\frac{0.2512}{0.0140}\right] = \arctan(17.94) \approx 86.8°$.

The response nearly lags the excitation by 90° — a hallmark of near-resonance.

181. Static deflection = 6.67 mm; Dynamic amplitude = 26.50 mm — amplification factor ≈ 3.97 ×. Despite meaningful damping ($\zeta = 12.65\%$), near-resonance still amplifies response by approximately 4 times. In practice, the machine operating speed should be shifted away from ω_n.

■

Example 6.5.8 — Torsional Vibration of an Engine Shaft

A circular steel shaft: $L = 1.2$ m, $d = 50$ mm. A flywheel with polar mass moment of inertia $J_d = 0.8$ kg·m² is attached at the free end; the other end is fixed. $G = 80$ GPa (shear modulus for steel).

182. Find the polar second moment of area. (b) Find torsional stiffness k_t. (c) Set up and solve the free torsional vibration equation. (d) Find natural frequency.

Solution:

$$J_{\text{shaft}} = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 6.136 \times 10^{-7} \text{ m}^4$$

$$k_t = \frac{G J_{\text{shaft}}}{L} = \frac{80 \times 10^9 \times 6.136 \times 10^{-7}}{1.2} = 40{,}907 \text{ N·m/rad}$$

183. Equation of motion for torsional vibration:

$$J_d \ddot{\theta} + k_t \theta = 0 \Longrightarrow \ddot{\theta} + \frac{k_t}{J_d}\theta = 0$$

This is the exact torsional analogue of the linear SHM equation $\ddot{x} + \omega_n^2 x = 0$.

General solution: $\theta(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$.

$$\omega_n = \sqrt{\frac{k_t}{J_d}} = \sqrt{\frac{40{,}907}{0.8}} = \sqrt{51{,}134} \approx 226.1 \text{ rad/s}$$

$$f_n = \frac{226.1}{2\pi} \approx 35.98 \text{ Hz} \approx 36 \text{ Hz} \qquad T = 1/f_n \approx 0.0278 \text{ s}$$

Engineering note: This frequency is well above typical engine excitation frequencies (engine at 3000 RPM produces 2nd-order excitation at 100 Hz), so resonance is not an issue in normal operating range.

■

Exercises 6.5.2 — Vibrations and Oscillations

6.5.6 A 12 kg mass on a spring ($k = 4800$ N/m) is displaced 40 mm and released with initial velocity 0.6 m/s. Find: (a) ω_n and T; (b) $x(t)$; (c) amplitude; (d) maximum velocity and maximum acceleration.

6.5.7 A structure has $m = 5000$ kg, $k = 2{,}000{,}000$ N/m, $c = 40{,}000$ N·s/m. (a) Find ω_n, ζ, and classify; (b) find ω_d; (c) write $x(t)$ for $x_0 = 0.03$ m, $v_0 = 0$.

6.5.8 Verify that $x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \varphi)$ satisfies $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ by direct substitution.

6.5.9 A machine operates at 1500 RPM. Mounting: $m = 80$ kg, $k = 200{,}000$ N/m, $\zeta = 0.07$. Unbalance force $F_0 = 600$ N. Find: (a) r; (b) DMF; (c) steady-state amplitude.

6.5.10 An undamped spring-mass system must resonate at exactly 8 Hz. If $m = 5$ kg, find: (a) required k; (b) amplitude if $x_0 = 0$, $v_0 = 0.5$ m/s; (c) jerk at $t = 0$.

6.5.11 For an underdamped system with $\delta = 0.80$ (logarithmic decrement), find ζ and the ratio of amplitudes after 5 complete oscillations.

Chapter 6 Summary

Section	Key Results and Formulas
5.1 Higher Derivatives	$(e^{ax})^{(n)} = a^n e^{ax}$ | $(\sin ax)^{(n)} = a^n \sin(ax + n\pi/2)$ | $(\ln x)^{(n)} = (-1)^{n-1}(n-1)!/x^n$
5.2 Physical Meaning	s' = velocity; s'' = acceleration; s''' = jerk. Speeding up ⇔ v and a same sign.
5.3 Concavity	$f'' > 0$: concave up; $f'' < 0$: concave down. Inflection where f'' changes sign.
5.3 Second Derivative Test	$f'(c) = 0$, $f''(c) > 0 \rightarrow$ local min; $f''(c) < 0 \rightarrow$ local max; $f''(c) = 0 \rightarrow$ inconclusive.
5.4 Leibniz Rule	$(fg)^{(n)} = \sum_k \binom{n}{k} f^{(k)} g^{(n-k)}$ — mirrors the Binomial Theorem.
5.5 Beam Deflection	$EI\, y^{(4)} = w(x)$. Simply supported + UDL: $\lvert y_{\max}\rvert = 5w_0L^4/(384EI)$.
5.5 Vibrations	$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$; $\zeta < 1$ underdamped; DMF $= 1/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$.

Chapter Review Problems

R1. Find the nth derivative of $f(x) = (2x + 3)^5$ without expanding.

R2. A particle's position is $s(t) = 2t^4 - 12t^3 + 18t^2$. Find intervals when: (a) particle moves in the positive direction; (b) speed is increasing.

R3. Find all inflection points of $f(x) = x^4 - 4x^3 + 6x^2$ and verify by sign analysis.

R4. Use the Leibniz rule to find $h^{(4)}(x)$ for $h(x) = x^3 e^{-x}$.

R5. A cantilever beam ($L = 3$ m, $EI = 5 \times 10^6$ N·m^2) carries 15 kN/m uniform load. Find maximum deflection and slope at the free end.

R6. A 60 kg mass is attached to a spring ($k = 24{,}000$ N/m) with $c = 720$ N·s/m. Find ω_n, ζ, ω_d, and write $x(t)$ for $x_0 = 0.03$ m, $v_0 = 0$.

R7. Derive from first principles the nth derivative of $f(x) = 1/(1 + x)$.

R8. A machine runs at 1200 RPM on a mounting with $m = 60$ kg, $k = 400{,}000$ N/m, $\zeta = 0.05$. Unbalance force $F_0 = 300$ N. Find the steady-state amplitude and comment on resonance risk.

R9. Prove that the fixed-fixed beam under UDL has exactly 5 times greater stiffness than the equivalent simply supported beam (compare maximum deflection formulas).

R10. Find $h^{(n)}(x)$ for $h(x) = e^{2x}\cos(3x)$. Express using $r = \sqrt{13}$ and $\theta = \arctan(3/2)$.

Selected Solutions to Exercises

Exercise 6.1.1

$$f(x) = 3x^7 - 2x^5 + x^3 - 4x + 9$$

$$f'(x) = 21x^6 - 10x^4 + 3x^2 - 4$$

$$f''(x) = 126x^5 - 40x^3 + 6x$$

$$f'''(x) = 630x^4 - 120x^2 + 6$$

$$f^{(4)}(x) = 2520x^3 - 240x$$

$$f^{(5)}(x) = 7560x^2 - 240$$

Exercise 6.2.1

$$h(t) = 80t^2 - 8/3\, t^3$$

184. $h'(t) = 160t - 8t^2$; $h'(4) = 640 - 128 = 512$ m/s.

185. $h''(t) = 160 - 16t$; $h''(6) = 160 - 96 = 64$ m/s^2.

186. $h''(t) = 0 \Rightarrow t = 10$ s; maximum velocity at $t = 10$: $h'(10) = 1600 - 800 = 800$ m/s.

187. $j(t) = h'''(t) = -16$ m/s^3 (constant — constant jerk implies linearly changing acceleration).

Exercise 6.3.1(a)

$f(x) = x^3 - 3x^2 + 4$; $f''(x) = 6x - 6 = 6(x-1)$.

$f'' < 0$ for $x < 1$ (concave down); $f'' > 0$ for $x > 1$ (concave up).

f'' changes sign at $x = 1$: inflection point at $\big(1,\ f(1)\big) = (1,\ 2)$.

Exercise 6.4.1(a)

$h(x) = x^2 e^x$. $f = x^2$, $g = e^x$; $f' = 2x$, $f'' = 2$, $f^{(k)} = 0$ for $k \geq 3$:

$$h^{(n)}(x) = \binom{n}{0} x^2 e^x + \binom{n}{1} 2x\, e^x + \binom{n}{2} 2\, e^x = e^x\big(x^2 + 2nx + n(n-1)\big)$$

Exercise 6.5.3

Simply supported beam, central point load P, span L. The deflection equation (from integration with appropriate BCs) gives:

$$|y_{\max}| = \frac{PL^3}{48EI}$$

Numerically: $P = 40{,}000$ N, $L = 6$ m, $EI = 10 \times 10^6$ N·m^2:

$$|y_{\max}| = \frac{40{,}000 \times 216}{48 \times 10{,}000{,}000} = \frac{8{,}640{,}000}{480{,}000{,}000} = 0.018 \text{ m} = 18 \text{ mm}$$

Exercise 6.5.6

$m = 12$ kg, $k = 4800$ N/m, $x_0 = 0.04$ m, $v_0 = 0.6$ m/s.

$$\omega_n = \sqrt{4800/12} = 20 \text{ rad/s} \qquad T = \pi/10 \approx 0.3142 \text{ s}$$

$A = x_0 = 0.04$ m; $B = v_0/\omega_n = 0.6/20 = 0.03$ m.

$$x(t) = 0.04\cos(20t) + 0.03\sin(20t) \text{ m}$$

$$C = \sqrt{0.04^2 + 0.03^2} = \sqrt{0.0025} = 0.05 \text{ m}$$

$$v_{\max} = C\omega_n = 0.05 \times 20 = 1.0 \text{ m/s} \qquad a_{\max} = C\omega_n^2 = 0.05 \times 400 = 20 \text{ m/s}^2$$

End of Chapter 6. This chapter has derived every result from first principles and demonstrated each with engineering-context worked examples. Solutions to all exercises are available in the Solutions Manual.

CHAPTER 7

Applications of Differentiation in Engineering

Optimisation • Curve Sketching • Related Rates • Newton-Raphson • Hessian

Chapter Introduction

Differentiation is far more than an abstract mathematical exercise. In engineering, the derivative is the language of change — it describes how systems evolve, how structures respond under load, how fluids accelerate through conduits, and how designs can be pushed towards their optimal performance. Chapter 7 takes the theoretical foundations of differential calculus developed in the preceding chapters and channels them squarely into engineering practice.

We begin in Section 7.1 with the classical optimization problem: given a mathematical model of a physical system, how do we identify the configuration that maximises benefit or minimises cost? The first and second derivative tests are rigorously derived and then applied to structural design, thermodynamic efficiency, and electrical power transmission. Section 7.2 extends this into the qualitative analysis of functions through systematic curve sketching — an indispensable skill when interpreting sensor outputs, signal responses, and system stability characteristics.

Section 7.3 introduces related rates, one of the most versatile techniques in applied calculus. Using the chain rule as its engine, related-rate analysis allows engineers to link the rates of change of two or more physical quantities. We study thermal expansion of solid bodies and the unsteady flow of fluids from tanks, both of which appear repeatedly in mechanical and chemical engineering.

Section 7.4 treats linear approximation and error analysis. Engineers never measure perfectly, and every computed quantity inherits uncertainty from its inputs. We derive the propagation-of-error formula from the differential of a function and apply it to measurement science and manufacturing tolerance analysis. Section 7.5 presents the Newton–Raphson method, deriving it from a truncated Taylor series and illustrating its remarkable quadratic convergence on transcendental and polynomial equations that arise in structural and fluid mechanics.

The chapter concludes in Section 7.6 by stepping into the multivariable world. The Hessian matrix — a two-dimensional analogue of the second derivative — is constructed,

its eigenvalue properties are interpreted physically, and it is used to classify saddle points, local minima, and local maxima in functions of two variables. An introduction to constrained optimisation using Lagrange multipliers provides the conceptual bridge to the optimisation theory encountered in advanced engineering design courses.

Throughout the chapter, every theoretical result is accompanied by original worked examples drawn from structural mechanics, heat transfer, fluid dynamics, electrical engineering, and manufacturing. Exercises at the end of each section challenge the reader to apply these ideas independently.

7.1 Optimisation in Engineering Design

7.1.1 The Calculus of Extreme Values

Engineering design is, at its heart, an optimisation problem. A structural engineer seeks the beam cross-section of greatest strength for a given weight. A chemical engineer minimises the surface area of a reactor vessel to reduce material cost while maintaining a specified volume. An electrical engineer maximises the power delivered to a load. In each case, a performance metric — a real-valued objective function — depends on one or more design variables, and the engineer's task is to identify the values of those variables that make the metric as large or as small as possible.

Let f be a function of a single design variable x, differentiable on an open interval. Recall that an interior point c is called a critical point of f whenever

$$\boldsymbol{f'(c) = 0 \text{ or } f'(c) \text{ does not exist.}} \quad (7.1)$$

At a critical point, the tangent to the graph of f is either horizontal or vertical (or absent), and it is precisely at such points that local extrema can occur. The following theorem, whose proof is a consequence of the mean-value theorem, furnishes the key classification tool.

Theorem / Result: First Derivative Test

Let c be a critical point of f, and suppose f is continuous on an open interval (a, b) containing c. Then:

- $If\ f'(x) > 0\ for\ a < x < c\ and\ f'(x) < 0\ for\ c < x < b$, then f has a local maximum at $x = c$.
- If $f'(x) < 0\ for\ a < x < c\ and\ f'(x) > 0\ for\ c < x < b$, then f has a local minimum at $x = c$.
- If f′ does not change sign at $x = c$, then f has neither a local maximum nor a local minimum at c.

When the second derivative exists at the critical point, a sharper test is available.

Theorem / Result: Second Derivative Test

Suppose f′(c) = 0 and f′′(c) exists. Then:

• $If\ f''(c) > 0$, then f has a strict local minimum at c (the curve is concave up).

• $If\ f''(c) < 0$, then f has a strict local maximum at c (the curve is concave down).

• $If\ f''(c) = 0$, the test is inconclusive; the first derivative test or higher-order analysis must be used.

The second derivative test has an elegant physical interpretation. $If\ f''(c) > 0$, the function behaves locally like a bowl opening upward — any small perturbation in the design variable increases the objective. If $f''(c) < 0$, the function is locally like an inverted bowl — small perturbations decrease performance. This geometric picture is invaluable when reviewing optimisation results for physical plausibility.

7.1.2 Derivation of the Second Derivative Test

We provide a concise derivation to establish familiarity with the underlying analysis. Suppose f′(c) = 0 and f′′(c) exists. By the definition of the second derivative:

$$\boldsymbol{f''(c) = lim[\Delta x \to 0]\ \frac{[\{f'(c + \Delta x) - f'(c)\}]}{\Delta x} = lim[\Delta x \to 0]\ \frac{[f'(c + \Delta x)]}{\Delta x}}$$

Since $f''(c) > 0$, there exists $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, the ratio $\frac{f'(x)}{x - c} > 0$. This means:

• For $x \in (c - \delta, c), we\ have\ x - c < 0, so\ f'(x) < 0.$

• For $x \in (c, c + \delta), we\ have\ x - c > 0, so\ f'(x) > 0.$

By the first derivative test, f has a strict local minimum at c. An identical argument with the signs reversed establishes the local maximum case when $f''(c) < 0$.

7.1.3 Optimisation Procedure

The following five-step procedure systematises single-variable optimisation problems in engineering.

1. Model: Identify the objective function f(x) to be maximised or minimised, and state any constraints.

2. Reduce: Use the constraint(s) to eliminate all but one design variable, obtaining f as a function of x alone.

3. Differentiate: Compute f′(x) and solve f′(x) = 0 to find all critical points.

4. Classify: Use the second derivative test (or first derivative test) to confirm whether each critical point is a maximum, minimum, or neither.

5. Check endpoints: Compare the critical-point value with the function values at any domain endpoints to determine the global extremum.

Worked Example 7.1.1: Optimal Cross-Section of a Structural Beam

A solid rectangular beam is to be cut from a circular log of radius r = 0.20 m. The bending strength S of the beam is proportional to wh^2, where w is the width and h is the depth of the cross-section. Find the dimensions w and h that maximise the bending strength.

Solution.

Step 1 — Model. The objective is to maximise $S = wh^2$.

Step 2 — Constraint. The rectangle must fit inside the circle of radius r, so the diagonal equals the diameter:

$$\boldsymbol{w^2 + h^2 = (2r)^2 = 4r^2 \Longrightarrow w = \sqrt{(4r^2 - h^2)}}$$

Substituting into S:

$$\boldsymbol{S(h) = h^2\sqrt{(4r^2 - h^2)}}$$

Step 3 — Differentiate.

$$\boldsymbol{\frac{dS}{dh} = 2h\sqrt{(4r^2 - h^2)} + h^2 \times \left[-\frac{h}{\sqrt{4r^2 - h^2}}\right]}$$

$$\boldsymbol{= h\frac{[\,2(4r^2 - h^2) - h^2]}{\sqrt{4r^2 - h^2}}}$$

$$\boldsymbol{= h\frac{8r^2 - 3h^2}{\sqrt{4r^2 - h^2}}}$$

Setting dS/dh = 0 (with h ≠ 0 and the denominator non-zero):

$$\boldsymbol{8r^2 - 3h^2 = 0 \Longrightarrow h^2 = \frac{8r^2}{3} \Longrightarrow h = r\sqrt{(\frac{8}{3})} = 2r\sqrt{6}\,/\,3}$$

Then:

$$\boldsymbol{w^2 = 4r^2 - h^2 = 4r^2 - \frac{8r^2}{3} = \frac{4r^2}{3} \Longrightarrow w = 2r\sqrt{3}\,/\,3}$$

Step 4 — Classify. The second derivative at the critical point is negative (the calculation is omitted here but follows readily), confirming a local maximum.

Step 5 — Numerical results with r = 0.20 m:

$$h = 2(0.20)\sqrt{6}/3 = 0.40 \times 0.8165 = 0.3266\,m$$

$$w = 2(0.20)\sqrt{3}/3 = 0.40 \times 0.5774 = 0.2309\,m$$

$$h/w = \sqrt{2} \approx 1.414\ (\textit{the optimal depth} - \textit{to} - \textit{width ratio})$$

This result — that the optimal depth-to-width ratio is √2 regardless of log radius — is a classical result in structural engineering. It shows that the optimal beam is always taller than it is wide, which is why standard I-beams and rectangular sections for floors have depths greater than their widths.

Worked Example 7.1.2: Minimum-Cost Cylindrical Storage Tank

A cylindrical storage tank with an open top must hold exactly V = 10 m³ of liquid chemical. The side wall costs \$8 per m² and the circular base costs \$12 per m². Find the radius r and height h that minimise total material cost, and compute that minimum cost.

Solution.

Volume constraint: πr²h = 10, so h = 10 / (πr²).

Total cost function:

$$C(r) = 8 \times (\textit{lateral area}) + 12 \times (\textit{base area})$$

$$C(r) = 8(2\pi rh) + 12(\pi r^2) = 16\pi rh + 12\pi r^2$$

Substituting h = 10/(πr²):

$$C(r) = 16\pi r \times 10/(\pi r^2) + 12\pi r^2 = 160/r + 12\pi r^2$$

Differentiating:

$$C'(r) = -160/r^2 + 24\pi r$$

Setting C′(r) = 0:

$$24\pi r = 160/r^2 \implies r^3 = 160/(24\pi) = 20/(3\pi)$$

$$r = \left[\frac{20}{3\pi}\right]^{\frac{1}{3}} = [2.1221\ldots]^{\frac{1}{3}} \approx 1.284\,m$$

The corresponding height:

$$h = 10/(\pi \times 1.284^2) = 10/(5.179) \approx 1.931\,m$$

Second derivative check:

$$C''(r) = 320/r^3 + 24\pi > 0\ for\ all\ r > 0 \implies confirmed\ minimum$$

Minimum cost:

$$C_{min} = 160/1.284 + 12\pi(1.284)^2 \approx 124.6 + 62.1 \approx \$186.7$$

$$\boldsymbol{r \approx 1.284\,m, h \approx 1.931\,m, C_min \approx \$186.70}$$

Notice that $h/r \approx 1.504 \approx 2(8/12) = 4/3$. The general result for a cost-optimised open-top cylinder is $\frac{h}{r} = \frac{2C_{side}}{C_{base}}$ which reduces to $h/r = 2$ when both surfaces cost the same per unit area — the classic equal-cost rule.

Worked Example 7.1.3: Maximum Power Transfer in an Electrical Circuit

A battery with electromotive force E (volts) and fixed internal resistance r (Ω) is connected to a variable external load resistance R (Ω). Derive the value of R that maximises the power P delivered to the load, and determine the maximum power.

Solution.

The current in the circuit is $I = E/(R + r)$. The power delivered to R is:

$$\boldsymbol{P(R) = I^2R = E^2R\,/\,(R + r)^2, R > 0}$$

Differentiating using the quotient rule:

$$\boldsymbol{dP/dR = E^2 \times \{\,(R + r)^2 - R \times 2(R + r)\,\}\,/\,(R + r)^4}$$

$$\boldsymbol{= E^2 \times (R + r - 2R)\,/\,(R + r)^3 = E^2(r - R)\,/\,(R + r)^3}$$

Setting $dP/dR = 0\ gives\ r - R = 0$, i.e.,

$$\boldsymbol{R = r\ (Maximum\ Power\ Transfer\ Theorem)}$$

Second derivative at R = r:

$$\boldsymbol{\frac{d^2P}{dR^2}_{\{R=r\}} = \frac{E^2(2R - 4r)}{R + r}^4|_{\{R=r\}} = \frac{E^2(2r - 4r)}{2r}^4 = -\frac{E^2}{8r^3} < 0}$$

Confirms a maximum. The maximum power delivered to the load is:

$$\boldsymbol{P_{max} = E^2r\,/\,(2r)^2 = E^2\,/\,(4r)}$$

This result — the maximum power transfer theorem — is fundamental in electrical engineering and is used in the design of audio amplifiers, antenna matching networks, and power converters. It states that exactly half the total generated power is wasted as heat in the internal resistance at maximum power transfer.

EXERCISES

1. A rectangular paddock is to be enclosed using 120 m of fencing, with one side along an existing stone wall (requiring no fencing). Find the dimensions of the paddock that maximise its area, and state the maximum area.

2. A manufacturer produces closed cylindrical tin cans of volume 500 cm³. The top and bottom discs cost twice as much per unit area as the side wall. Derive and solve the optimisation problem to find the radius and height that minimise total cost. Show that the optimal ratio h/r = 1.

3. A power cable connects a generator on one bank of a 300 m wide river to a substation located 600 m downstream on the opposite bank. Underwater cable costs \$60/m and overland cable costs \$35/m. Let x be the horizontal distance downstream of the crossing point (measured along the far bank). Write P(x), the total cable cost as a function of x, differentiate, and find the optimal x. [Ans: x ≈ 265.1 m]

4. The deflection of a simply supported beam under a point load F at distance x from one end (beam length L) is proportional to $\delta(x) = Fx(L^2 - x^2)/(6EIL)$ for 0 ≤ x ≤ L. Find the value of x (in terms of L) at which maximum deflection occurs. [Ans: x = L/√3]

5. A cylindrical boiler shell (closed both ends) is to enclose a fixed volume V. The hemispherical end-caps are made from material twice as costly per unit area as the cylindrical body. Find the optimal radius-to-height ratio for minimum cost. Show that h = 2r when end-cap and body materials cost the same.

6. An operational amplifier in a transimpedance configuration has gain $G = R_f/(1 + R_f/R_{in})$. Treating R_f as the design variable (with R_{in} fixed), find the value of R_f that maximises the rate of change dG/dR_f , and interpret the result physically.

7.2 Curve Sketching and System Behaviour

7.2.1 Why Curve Sketching Matters in Engineering

Engineers rely on graphical representations not merely for presentation but as analytical tools in their own right. A well-executed sketch of a displacement–time curve reveals whether a vibrating component overshoots its equilibrium (overshoot analysis), whether a control output saturates, or whether a chemical concentration approaches a safe steady state asymptotically or oscillates through it. Systematic curve sketching using calculus provides far more information than numerical tabulation alone, because it makes the qualitative structure of a function visible.

7.2.2 Systematic Procedure for Curve Sketching

For a function f(x), the following eight-step procedure produces a reliable sketch and, more importantly, a thorough analytical characterisation.

1. Domain. Determine the natural domain of f (values of x where f is defined).

2. Intercepts. Compute f(0) (y-intercept). Solve f(x) = 0 for x-intercepts where tractable.

3. Symmetry. Check if f(−x) = f(x) (even, symmetric about y-axis) or f(−x) = −f(x) (odd, symmetric about origin). Identify periodicity if relevant.

4. Asymptotes. Compute lim_{x→±∞} f(x) for horizontal asymptotes. Identify vertical asymptotes at points where f → ±∞.

5. First derivative. Compute f′(x). Find all critical points. Determine intervals of increase (f′ > 0) and decrease (f′ < 0).

6. Local extrema. Apply the second derivative test (or first derivative test) at each critical point.

7. Second derivative. Compute f′′(x). Determine intervals of concavity: concave up where f′′ > 0, concave down where f′′ < 0.

8. inflection points. Find all x where f′′(x) = 0 and the second derivative changes sign.

7.2.3 Concavity and Points of inflection

The second derivative characterises the curvature of a graph. When f′′(x) > 0, the slope f′(x) is increasing — the curve is bending upward ('concave up' or 'convex'). When f′′(x) < 0, the slope is decreasing — the curve bends downward ('concave down' or 'concave'). An inflection point (x_i, f(x_i)) is a point where the concavity changes, meaning f′′(x_i) = 0 and f′′ changes sign at x_i.

In engineering, inflection points have specific physical meanings. In a loading curve, they mark the transition from hardening to softening behaviour. In a step response, they indicate maximum rate of change of the output. In population dynamics and epidemic models, they mark the peak rate of growth.

Worked Example 7.2.1: Analysis of a Damped Transient Response

In electrical and mechanical engineering, the transient response of a first-order damped system to an impulse or initial condition is modelled by the function

f(t) = t e^{−αt}, t ≥ 0, α > 0

where t is time and α is the damping coefficient (s^{-1}). Perform a complete curve analysis and identify all key features.

Solution.

Step 1: Domain is [0, ∞). Step 2: f(0) = 0 (origin), f(t) = 0 only at t = 0 since e^{−αt} > 0 for all t. Step 3: No symmetry (t ≥ 0 only). Step 4: lim_{t→∞} te^{−αt} = 0 (exponential beats polynomial), so y = 0 is a horizontal asymptote.

Step 5 — First derivative (product rule):

f′(t) = e^{−αt} + t(−α)e^{−αt} = e^{−αt}(1 − αt)

Since e^{−αt} > 0 always, the sign of f′ is determined by (1 − αt).

• f′(t) > 0 for t < 1/α — function is increasing.

• f′(t) < 0 for t > 1/α — function is decreasing.

• Critical point: t_c = 1/α with f(1/α) = (1/α)e^{− 1} = 1/(αe). This is the peak amplitude.

Step 7 — Second derivative:

f′′(t) = −αe^{−αt}(1−αt) + e^{−αt}(−α) = αe^{−αt}(αt − 2)

Second derivative test: f′′(1/α) = αe^{− 1}(α/α − 2) = αe^{− 1}(− 1) = −α/e < 0, confirming a local (and global) maximum.

Step 8 — inflection point: f′′(t) = 0 when αt − 2 = 0, so t_i = 2/α.

• Concave down on (0, 2/α) — rate of decrease is accelerating.

• Concave up on (2/α, ∞) — rate of decrease is slowing.

Peak at t = 1/α, amplitude = 1/(αe); inflection at t = 2/α

The ratio $\frac{t_i}{t_c} = 2$ is independent of α. Engineers use this fact when tuning instrument calibration: the point at which the sensor output begins to 'flatten out' always occurs at exactly twice the time of the peak reading, regardless of the system's specific damping constant.

Worked Example 7.2.2: Curve Analysis of a Dimensionless Flow Coefficient

In pipe flow analysis, a dimensionless flow coefficient takes the form

f(x) = (x² − 4) / (x² + 4), x ∈ ℝ

where x represents a normalised velocity. Perform a complete curve sketch analysis.

Solution.

Domain: All real numbers. No vertical asymptotes since x² + 4 ≥ 4 > 0.

Intercepts: f(0) = −4/4 = −1. Setting f(x) = 0: x² − 4 = 0, so x = ±2.

Symmetry: f(−x) = (x² − 4)/(x² + 4) = f(x) — even function, symmetric about the y-axis.

Asymptotes: $lim_{\{x\to\pm\infty\}}\, f(x) = lim\,(1 - 4/x^2)/(1 + 4/x^2) = 1$. Horizontal asymptote y = 1 (but f < 1 for all finite x).

First derivative:

f′(x) = [2x(x²+4) − (x²−4)(2x)] / (x²+4)²

= 2x[(x²+4) − (x²−4)] / (x²+4)² = 2x(8) / (x²+4)² = 16x / (x²+4)²

Critical point: x = 0, with f(0) = −1. f′(x) < 0 for x < 0, f′(x) > 0 for x > 0, so this is a global minimum.

Second derivative:

$$\mathbf{f''(x) = d/dx\,[\,16x/(x^2+4)^2\,]}$$

$$\mathbf{= 16(x^2+4)^2 - 16x \times 2(x^2+4)(2x)] / (x^2+4)^4}$$

$$\mathbf{= 16(x^2+4)[(x^2+4) - 4x^2] / (x^2+4)^4 = 16(4 - 3x^2) / (x^2+4)^3}$$

inflection points: $4 - 3x^2 = 0 \Longrightarrow x = \pm 2/\sqrt{3} = \pm 2\sqrt{3}/3 \approx \pm 1.155$

$f(\pm 2\sqrt{3}/3) = (4/3 - 4)/(4/3 + 4) = (-8/3)/(16/3) = -1/2$

Summary of curve features: global minimum at (0, −1); inflection points at $(\pm 2\sqrt{3}/3, -1/2)$; approaches asymptote y = 1 from below as $x \to \pm\infty$; crosses x-axis at x = ±2. The function is even, so the entire sketch is symmetric about the y-axis. This S-shaped curve is characteristic of many normalised transfer functions in control engineering.

EXERCISES

1. Perform a complete curve sketch analysis of $f(x) = x^4 - 8x^2 + 5$. Identify all critical points, inflection points, and intervals of concavity.

2. Analyse f(t) = e^{−t} sin(2t) for t ≥ 0. Find the first peak amplitude and its time of occurrence, and identify the first two inflection points. [Hint: express f′ as a product and use the identity for the angle sum.]

3. For the function f(x) = ln(x)/x (x > 0), determine the domain, find and classify the single critical point, and identify all asymptotic behaviour. Sketch the curve.

4. The tension in a cable supporting a suspended load varies as $T(x) = T_0 \sqrt{(1 + \sinh^2(x/a))}$ for a catenary cable. Using the substitution u = x/a, analyse the concavity of T(x) and explain its physical significance for cable design.

5. A dimensionless drag coefficient is given by $C_D(x) = (x^3 - 3x) / (x^2 + 2)$. Find all local extrema and inflection points. Determine the horizontal asymptote and sketch the general shape.

7.3 Related Rates in Physical Systems

7.3.1 The Chain Rule as the Engine of Related Rates

In many physical systems, several quantities change simultaneously with time, and these quantities are linked by a geometric or physical constraint. Related-rate problems exploit the chain rule to relate the time derivatives of constrained quantities.

The fundamental principle is straightforward: if two quantities A and B are related by an equation E(A, B) = 0, and both A and B are differentiable functions of time t, then differentiating E implicitly with respect to t yields a relationship between dA/dt and dB/dt. Knowing one rate allows the engineer to compute the other.

Formally, suppose y = f(u) and u = g(t). Then by the chain rule:

$$\mathbf{dy/dt = (dy/du)(du/dt) = f'(u) \times g'(t)}$$

The procedure for solving related-rates problems is:

1. Draw a diagram and introduce clearly defined variables for all relevant quantities.

2. Write the geometric or physical equation relating the variables.

3. Differentiate both sides with respect to time t (using the chain rule for all terms).

4. Substitute the known rates and values at the instant of interest, and solve for the unknown rate.

7.3.2 Expanding Bodies

When solid bodies are heated, they expand. This phenomenon — thermal expansion — provides a rich set of related-rates problems because the three-dimensional geometry of the object constrains how its different dimensions change. The governing equation for isotropic (equal in all directions) thermal expansion is:

$$\mathbf{L(T) = L_0 (1 + \alpha_L \Delta T)}$$

where L_0 is the original length, α_L is the linear thermal expansion coefficient (K^{-1}), $\Delta T = T - T_0$ is the temperature change, and L(T) is the length at temperature T. Differentiating with respect to time:

$$\mathbf{dL/dt = L_0 \alpha_L (dT/dt)}$$

For volumetric expansion, the volumetric coefficient $\alpha_V \approx 3\alpha_L$ (for isotropic materials), giving:

$$\mathbf{V(T) = V_0(1 + \alpha_V \Delta T) \Longrightarrow dV/dt = V_0 \alpha_V (dT/dt)}$$

Worked Example 7.3.1: Thermal Expansion of a Spherical Pressure Vessel

A spherical steel pressure vessel has radius r = 0.50 m at ambient temperature. The vessel is heated such that its radius increases at a rate of dr/dt = 0.004 cm/s = 4×10^{-5} m/s. Find: (a) the rate of increase of the volume, and (b) the rate of increase of the surface area, at the instant described.

Solution.

(a) Volume of sphere: $V = (4/3)\pi r^3$. Differentiating implicitly with respect to time:

$dV/dt = 4\pi r^2 (dr/dt)$

Substituting r = 0.50 m and dr/dt = 4×10^{-5} m/s:

$dV/dt = 4\pi(0.50)^2(4 \times 10^{-5}) = 4\pi(0.25)(4 \times 10^{-5}) = 1.257 \times 10^{-4}\ m^3/s$

(b) Surface area: $S = 4\pi r^2$. Differentiating:

$dS/dt = 8\pi r\,(dr/dt) = 8\pi(0.50)(4 \times 10^{-5}) = 5.027 \times 10^{-4}\ m^2/s$

$dV/dt \approx 1.257 \times 10^{-4}\ m^3/s;\ dS/dt \approx 5.027 \times 10^{-4}\ m^2/s$

Note that dS/dt = 2(dV/dt)/r, a relation that follows directly from the formula dV/dr = r/2 × dS/dr for spheres. This ratio is important in heat transfer: the rate of heat loss through the surface grows at 2/r times the volumetric rate of expansion — as vessels shrink (smaller r), surface effects become increasingly important relative to volumetric effects.

Worked Example 7.3.2: Thermal Expansion of a Rectangular Aluminium Plate

A rectangular aluminium plate has initial dimensions $L_0 = 25\ cm$ and $W_0 = 18\ cm$. The plate is heated uniformly, and the temperature rises at a rate of dT/dt = 4°C/s. The linear thermal expansion coefficient of aluminium is $\alpha_L = 23 \times 10^{-6}\ K^{-1}$. Find the rates of change of (a) the length, (b) the width, and (c) the area of the plate at the moment when the temperature has risen by ΔT = 0 (i.e., at the initial instant).

Solution.

The length and width as functions of temperature:

$$\boldsymbol{L(T) = L_0(1 + \alpha_L\ T), W(T) = W_0(1 + \alpha_L\, T)}$$

(where T is measured from the reference temperature, so ΔT = T).

(a) $dL/dt = L_0\,\alpha_L\,(dT/dt) = 0.25 \times 23 \times 10^{-6} \times 4 = 2.30 \times 10^{-5}\,m/s = 0.0230\,mm/s$

(b) $dW/dt = W_0\,\alpha_L\,(dT/dt) = 0.18 \times 23 \times 10^{-6} \times 4 = 1.656 \times 10^{-5}\,m/s = 0.01656\,mm/s$

(c) $Area\ A = LW. Differentiating\ with\ the\ product\ rule$:

$$\boldsymbol{dA/dt = W(dL/dt) + L(dW/dt)}$$

$At\ the\ initial\ instant, L = L_0 = 0.25\,m\ and\ W = W_0 = 0.18\,m$:

$$\boldsymbol{dA/dt = 0.18(2.30 \times 10^{-5}) + 0.25(1.656 \times 10^{-5})}$$

$$\boldsymbol{= 4.14 \times 10^{-6} + 4.14 \times 10^{-6} = 8.28 \times 10^{-6}\,m^2/s}$$

Verification using $A = L_0 W_0 (1 + \alpha_L T)^2$:

$$\boldsymbol{dA/dt = L_0 W_0 \times 2(1 + \alpha_L T)\alpha_L \left(\frac{dT}{dt}\right)|_{\{T=0\}} = (0.25)(0.18)(2)(23 \times 10^{-6})(4)}$$
$$\boldsymbol{= 8.28 \times 10^{-6}\, m^2/s \checkmark}$$
$$\boldsymbol{dA/dt = 8.28 \times 10^{-6}\, m^2/s \approx 0.828\, mm^2/s}$$

7.3.3 Fluid Flow — Torricelli's Theorem and Tank Drainage

One of the most important related-rates applications in hydraulic and chemical engineering is the unsteady draining of a tank. When a liquid exits through an orifice at the base of a tank under gravity alone, the exit velocity is governed by Torricelli's theorem, which itself follows from Bernoulli's energy equation under the assumption that the liquid surface drops slowly compared with the exit jet:

$$\boldsymbol{v_{\{exit\}} = \sqrt{(2gh)}}$$

where h is the current depth of liquid above the orifice and g = 9.81 m/s² is gravitational acceleration. If the orifice has cross-sectional area a, the volumetric outflow rate is:

$$\boldsymbol{Q_{\{out\}} = a \times v_\{exit\} = a\sqrt{(2gh)}}$$

The continuity equation for the control volume of the tank states that the rate of decrease of stored volume equals the outflow rate (assuming no inflow):

$$\boldsymbol{dV/dt = -Q_{\{out\}} = -a\sqrt{(2gh)}}$$

For a tank whose cross-sectional area A(h) depends on the fill level h, the volume element satisfies $dV = A(h)\, dh, so$:

$$\boldsymbol{A(h)\, dh/dt = -a\sqrt{(2gh)}}$$

This is the key differential equation for unsteady tank drainage. Once A(h) is known (from the tank geometry), we can find dh/dt — the rate at which the liquid level drops — at any given instant.

Worked Example 7.3.3: Drainage Rate from a Conical Tank

A conical tank is positioned with its vertex at the bottom and its axis vertical. The tank has a half-angle θ = 30° (the half-angle between the axis and the slant side). A circular drainage hole of radius $r_h = 0.008\, m$ is located at the vertex. At a certain instant, the water depth is h = 3.0 m. Determine the rate at which the water level is falling at that instant.

Solution.

The radius of the water surface at depth h: $R = h\, tan(30°) = h/\sqrt{3}$.

Cross-sectional area of water surface: $A(h) = \pi R^2 = \pi h^2/3$.

Drainage orifice area: $a = \pi r_h^2 = \pi(0.008)^2 = 2.011 \times 10^{-4}\, m^2$.

Exit velocity at h = 3.0 m (Torricelli):

$$\boldsymbol{v_{\{exit\}} = \sqrt{(2 \times 9.81 \times 3.0)} = \sqrt{(58.86)} = 7.672\, m/s}$$

Outflow rate: $Q = a \times v_{\{exit\}} = 2.011 \times 10^{-4} \times 7.672 = 1.543 \times 10^{-3}\, m^3/s$

Applying the continuity equation A(h) dh/dt = −Q:

$$\boldsymbol{(\pi \times 3.0^2/3) \times dh/dt = -1.543 \times 10^{-3}}$$

$$\boldsymbol{(3\pi) \times dh/dt = -1.543 \times 10^{-3}}$$

$$\boldsymbol{dh/dt = -1.543 \times 10^{-3} / (3\pi) = -1.637 \times 10^{-4}\, m/s}$$

$$\boldsymbol{dh/dt \approx -1.637 \times 10^{-4}\, m/s\ (level\ dropping\ at \approx 0.164\, mm/s)}$$

Notice that as h decreases, the outflow rate Q decreases (via Torricelli), while the water surface area A(h) also decreases. The rate |dh/dt| is not constant, and the full solution for h(t) requires solving the resulting separable ODE — a topic covered in a differential equations course.

Worked Example 7.3.4: Level Change in a Pump-Drain System

A cylindrical water tank of radius R = 2.5 m is simultaneously filled by a pump at a constant rate of $Q_{\{in\}} = = 0.4\, m^3/min$ and drained through a valve whose flow resistance gives $Q_{\{out\}} = k\sqrt{h}, where\ k = 0.15\, m^{\{\frac{5}{2}\}}/min$. Find dh/dt when h = 1.6 m.

Solution.

Cross-sectional area of tank: $A = \pi R^2 = \pi(2.5)^2 = 19.635\, m^2$

Net volumetric rate: $dV/dt = Q_{\{in\}} - Q_{\{out\}} = 0.4 - 0.15\sqrt{h}$

Since $\frac{dV}{dt} = A\frac{dh}{dt}$:

$$\boldsymbol{19.635 \times dh/dt = 0.4 - 0.15\sqrt{(1.6)}}$$

$$\boldsymbol{19.635 \times dh/dt = 0.4 - 0.15(1.2649) = 0.4 - 0.1897 = 0.2103}$$

$$\boldsymbol{dh/dt = 0.2103 / 19.635 = 0.01071\, m/min}$$

$$\boldsymbol{dh/dt \approx 0.01071\, m/min\ (level\ rising\ at \approx 10.7\, mm/min)}$$

Steady-state level (when dh/dt = 0):

$$\boldsymbol{0.4 = 0.15\sqrt{h_\infty} \Longrightarrow \sqrt{h_\infty} = 0.4/0.15 = 8/3 \Longrightarrow h_\infty = 64/9 \approx 7.11\, m}$$

The steady-state level $h_\infty = 7.11\ m$ is approached asymptotically. At h = 1.6 m, the tank is filling, and dh/dt > 0, confirming the system is far from steady state. This type of analysis is fundamental in chemical plant design for sizing holding tanks and controlling overflow risks.

EXERCISES

1. A spherical helium balloon is inflated so that its volume increases at 6 cm³/s. Find the rate of increase of (a) the radius and (b) the surface area when r = 8 cm.

2. Water is poured into an inverted conical tank (vertex at top, axis vertical) with half-angle 45° at 2.5 m³/min. Find the rate at which the water depth increases when h = 3.0 m below the vertex. [Ans: dh/dt = 2.5/(9π) ≈ 0.0884 m/min]

3. Two currents flow in series through a resistor network. Resistor $R_1 = 20\,\Omega$ is increasing at 3 Ω/s while resistor $R_2 = 15\,\Omega$ is decreasing at 1 Ω/s. The total resistance is $R = R_1 + R_2$. Find dR/dt.

4. A ladder 8 m long leans against a vertical wall. Its base slides away at 0.5 m/s. Find the rate at which the top of the ladder descends when the base is 4 m from the wall. Verify that the rate tends to infinity as the ladder approaches horizontal, and explain the engineering significance.

5. A rectangular channel has width w = 3 m. Water enters at 1.2 m³/s and the outflow rate (over a weir) is $Q_{\{out\}} = 0.8\,h^{\{\frac{3}{2}\}}\ m^3/s$. Find dh/dt when h = 0.5 m, and find the equilibrium depth h_∞.

6. The power P = V²/R is dissipated in a resistor. At a certain instant, V = 24 V and is falling at 0.3 V/s, while R = 6 Ω and is rising at 0.04 Ω/s. Find dP/dt at that instant.

7.4 Linear Approximation and Error Analysis

7.4.1 The Linearisation of a Function

Engineers seldom work with exact values. Every measurement carries an error, every material property has a tolerance, and every model contains approximations. Understanding how small uncertainties in inputs propagate through calculations to produce uncertainties in outputs is a core skill in engineering analysis. Differential calculus provides the precise mathematical tool for this analysis: the linear approximation.

Consider a differentiable function f(x). The tangent line to the graph of f at a reference point x = a has equation:

$$\boldsymbol{L(x) = f(a) + f'(a)(x - a)}$$

The function L(x) is the linearisation (or first-order Taylor approximation) of f about x = a. When x is close to a, the difference |x − a| = |Δx| is small, and L(x) is an excellent approximation to f(x). More precisely, using the Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2 + \ldots$$

the linearisation retains only the first-order term. The error in the approximation is of second order in Δx:

$$f(x) - L(x) = f''(a)(\Delta x)^2/2 + O((\Delta x)^3)$$

This shows that for small perturbations Δx, the linearisation error is negligible to first order — a principle that underpins classical error analysis.

7.4.2 The Differential and Error Propagation

The differential of f at x is defined as:

$$dy = f'(x)\,dx$$

Here, dx is any increment in x, and dy is the corresponding increment in the linear approximation of f. In engineering practice, if x is a measured quantity with an absolute uncertainty δx, the propagated uncertainty in y = f(x) is approximated by:

$$\delta y \approx |f'(x)|\,\delta x$$

The absolute value is used because uncertainty is always positive regardless of the sign of f′. The relative uncertainty is:

$$\delta y/|y| \approx |f'(x)/f(x)|\,\delta x = |d(\ln f)/dx|\,\delta x$$

This formula has a beautiful logarithmic interpretation: the relative error in y equals the relative error in x multiplied by the absolute value of the logarithmic derivative of f.

7.4.3 Measurement Uncertainty — Functions of Multiple Variables

When $y = f(x_1, x_2, \ldots, x_n)$ *and each* x_i is measured independently with uncertainty δx_i, the worst-case (maximum) propagated uncertainty in y is given by the total differential:

$$\delta y_{\{max\}} = \left|\frac{\partial f}{\partial x_1}\right| \delta x_1 + \left|\frac{\partial f}{\partial x_2}\right| \delta x_2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right| \delta x_n$$

This is the worst-case method: it assumes all errors add constructively. A more realistic estimate, appropriate when errors are random and independent, is the root-sum-squared (RSS) method:

$$\delta y_{\{RSS\}} = \sqrt{[\,(\frac{\partial f}{\partial x_1}\delta x_1)^2 + (\frac{\partial f}{\partial x_2}\delta x_2)^2 + \cdots + (\frac{\partial f}{\partial x_n}\delta x_n)^2\,]}$$

The RSS method is standard in precision metrology and is the basis for the GUM (Guide to the Expression of Uncertainty in Measurement), the international standard for uncertainty reporting.

Worked Example 7.4.1: Linearisation of the Pendulum Period

The period of a simple pendulum of length L is T = 2π√(L/g). A pendulum designed for a period of $T_0 = 2.000\ s$ at reference length $L_0 = g/\pi^2 \approx 0.9933\ m$ (with g = 9.81 m/s²) is manufactured with a length error of δL = +5 mm. Estimate the resulting error in the period using linear approximation.

Solution.

Reference period check: $T_0 = 2\pi\sqrt{(0.9933/9.81)} = 2\pi\sqrt{(0.10125)} = 2\pi(0.31820) = 2.000\ s$ ✓

Derivative:

$$\boldsymbol{dT/dL = 2\pi \times 1/(2\sqrt{(Lg)}) = \pi/\sqrt{(Lg)}}$$

At $L_0 = 0.9933\ m$:

$$\boldsymbol{dT/dL|_{\{L_0\}} = \pi/\sqrt{(0.9933 \times 9.81)} = \pi/\sqrt{(9.745)} = \pi/3.122 = 1.006\ s/m}$$

Linear approximation for the period error:

$$\boldsymbol{\delta T \approx |dT/dL| \times \delta L = 1.006 \times 0.005 = 0.00503\ s}$$

Relative error:

$$\boldsymbol{\delta T/T_0 = 0.00503/2.000 = 0.00252 = 0.252\%}$$

A 5 mm error in length produces ≈ 0.00503 s (≈ 0.25%) error in period

Exact verification: T(1.0 m - but using $L_0 + 0.005$):

$$\boldsymbol{T(0.9983) = 2\pi\sqrt{(0.9983/9.81)} = 2\pi\sqrt{(0.10177)} = 2\pi(0.31902) = 2.00516\ s}$$

Exact δT = 2.00516 − 2.000 = 0.00516 s. Approximation error: 0.00516 − 0.00503 = 0.00013 s (< 3%) — excellent accuracy.

Worked Example 7.4.2: Error Propagation in Electrical Resistance Measurement

The electrical resistance of a cylindrical copper wire is R = ρL/(πr²), where ρ = 1.68 × 10⁻⁸ Ω·m (resistivity), L is wire length, and r is wire radius. A 10 m length of wire has radius r = 0.001 m. Measurements have uncertainties δL = 0.01 m and δr = 5 × 10⁻⁵ m. Compute the nominal resistance, the absolute uncertainty using (a) the worst-case method and (b) the RSS method, and identify which measurement contributes most to the total error.

Solution.

Nominal resistance:

$$R = (1.68 \times 10^{-8} \times 10) / (\pi \times 10^{-6}) = 1.68 \times 10^{-7} / (3.1416 \times 10^{-6}) = 0.05347\ \Omega$$

Partial derivatives:

$$\partial R/\partial L = \rho/(\pi r^2) = 1.68 \times 10^{-8}/(3.1416 \times 10^{-6}) = 5.347 \times 10^{-3}\ \Omega/m$$

$$|\partial R/\partial r| = 2\rho L/(\pi r^3) = 2 \times 1.68 \times 10^{-8} \times 10/(3.1416 \times 10^{-9}) = 3.36 \times 10^{-7}/(3.1416 \times 10^{-9}) = 106.9\ \Omega/m$$

Error contributions:

$$|\partial R/\partial L|\delta L = 5.347 \times 10^{-3} \times 0.01 = 5.347 \times 10^{-5}\ \Omega$$

$$|\partial R/\partial r|\delta r = 106.9 \times 5 \times 10^{-5} = 5.347 \times 10^{-3}\ \Omega$$

(a) Worst-case uncertainty:

$$\delta R_{\{max\}} = 5.347 \times 10^{-5} + 5.347 \times 10^{-3} \approx 5.40 \times 10^{-3}\ \Omega$$

(b) RSS uncertainty:

$$\delta R_{\{RSS\}} = \sqrt{[(5.347 \times 10^{-5})^2 + (5.347 \times 10^{-3})^2]}$$

$$\approx \sqrt{[2.859 \times 10^{-9} + 2.859 \times 10^{-5}]} \approx \sqrt{(2.860 \times 10^{-5})} \approx 5.348 \times 10^{-3}\ \Omega$$

$$R = 0.05347\ \Omega \pm 5.35 \times 10^{-3}\ \Omega\ (relative\ uncertainty \approx 10.0\%)$$

The radius measurement error ($5.35 \times 10^{-3}\ \Omega$) dominates the length measurement error ($5.35 \times 10^{-5}\ \Omega$) by a factor of 100. This occurs because $R \propto r^{-2}$, so the relative error in R is twice the relative error in r. Doubling the radius measurement precision would reduce total uncertainty by approximately 99%, while halving the length error would have negligible effect. This analysis guides the engineer's instrument selection and measurement protocol.

7.4.3 Tolerance Analysis in Manufacturing

In manufacturing, every component is specified with a nominal (target) dimension and a tolerance — the permissible deviation from the nominal. When components are assembled, the tolerances stack up, and the designer must ensure that the assembled product meets its functional requirements even in the worst case.

For an assembly parameter $Y = f(X_1, X_2, \ldots, X_n)$ where each component dimension X_i *has nominal value* μ_i *and bilateral tolerance* t_i (*meaning the actual value lies in* $[\mu_i - t_i, \mu_i + t_i]$):

Worst-case tolerance: $t_Y = \Sigma \left|\frac{\partial f}{\partial X_i}\right| t_i$

Statistical (RSS) tolerance: $t_Y^{\{RSS\}} = \sqrt{(\Sigma \left(\frac{\partial f}{\partial X_i}\right)^2 t_i^2)}$

Worked Example 7.4.3: Shaft-Bearing Clearance Tolerance

A precision shaft of nominal diameter d = 40.00 mm with bilateral tolerance $t_d = \pm 0.025$ mm is to fit inside a bearing bore of nominal diameter D = 40.08 mm with tolerance t_D = ±0.020 mm. The functional clearance is C = D – d. Calculate: (a) the nominal clearance, (b) the worst-case maximum and minimum clearances, (c) the worst-case bilateral tolerance, and (d) the statistical tolerance using the RSS method.

Solution.

(a) Nominal clearance: $C_{\{nom\}} = 40.08 - 40.00 = 0.08\, mm$

(b) Maximum clearance (largest bore, smallest shaft):

$$\boldsymbol{C_{\{max\}} = (D + t_D) - (d - t_d) = (40.08 + 0.020) - (40.00 - 0.025)}$$
$$\boldsymbol{= 40.100 - 39.975 = 0.125\, mm}$$

Minimum clearance (smallest bore, largest shaft):

$$\boldsymbol{C_{\{min\}} = (D - t_D) - (d + t_d) = (40.08 - 0.020) - (40.00 + 0.025)}$$
$$\boldsymbol{= 40.060 - 40.025 = 0.035\, mm}$$

(c) Worst-case tolerance: $\partial C/\partial D = +1, \partial C/\partial d = -1$

$$\boldsymbol{t_{\{C,WC\}} = |+1| \times t_D + |-1| \times t_d = 0.020 + 0.025 = 0.045\, mm}$$

So C = 0.080 ± 0.045 mm (worst case), confirming (b): 0.080 ± 0.045 ⟹ [0.035, 0.125] ✓

(d) RSS tolerance:

$$\boldsymbol{t_{\{C,RSS\}} = \sqrt{(t_D^2 + t_d^2)} = \sqrt{(0.020^2 + 0.025^2)} = \sqrt{(0.000400 + 0.000625)}}$$
$$\boldsymbol{= \sqrt{(0.001025)} = 0.032\, mm}$$

$$\boldsymbol{C = 0.080 \pm 0.045\, mm\ (worst\ case)\ or\ 0.080 \pm 0.032\, mm\ (RSS\ statistical)}$$

The RSS method gives a tighter tolerance (0.032 vs 0.045 mm) because it recognises that both dimensions being simultaneously at their extremes is statistically unlikely. In high-volume production, the RSS method allows designers to relax individual tolerances (reducing machining cost) while still ensuring acceptable assembly quality with high probability. The choice between worst-case and RSS depends on the consequence of failure and the required reliability level.

EXERCISES

1. Use linearisation to estimate (a) $\sqrt{(24.5)}$ using the reference point a = 25, and (b) $(1.03)^5$ using a = 1. In each case, compare with the calculator value and compute the percentage error.

2. The volume of a sphere is $V = (4/3)\pi r^3$. If the radius is measured as r = 5 cm with an uncertainty of ±0.04 cm, find the (a) absolute uncertainty in V and (b) relative percentage uncertainty.

3. The kinetic energy of a rotating flywheel is $E = (1/2)I\omega^2$, where $I = 0.8 \text{ kg·m}^2 \pm 0.01 \text{ kg·m}^2$ and $\omega = 120 \text{ rad/s} \pm 2 \text{ rad/s}$. Find the worst-case and RSS uncertainties in E. Which variable contributes more to the uncertainty?

4. A rectangular concrete beam (width b, height h, span L) has mid-span deflection $\delta = 5wL^4/(384EI)$ where $I = bh^3/12$. If $b = 0.20 \pm 0.003$ m, $h = 0.35 \pm 0.004$ m, find the percentage uncertainty in I using the worst-case method.

5. Three resistors $R_1 = 100 \pm 2\,\Omega, R_2 = 150 \pm 3\,\Omega, R_3 = 200 \pm 4\,\Omega$ are connected in parallel. The total resistance is $1/R = 1/R_1 + 1/R_2 + 1/R_3$. Find the nominal R and its worst-case uncertainty using partial derivatives.

7.5 Newton–Raphson Method

7.5.1 Motivation and Derivation

A great many engineering equations cannot be solved in closed form. The critical buckling load of a complex structural member, the exit Mach number through a converging-diverging nozzle, the friction factor in turbulent pipe flow, the roots of a characteristic polynomial for a vibrating system — all require numerical solution. The Newton–Raphson method is the most widely used and powerful iterative root-finding algorithm in engineering practice. It is derived directly from the concept of linearisation.

Suppose we seek a root x* of the equation f(x) = 0, where f is differentiable near x*. Beginning with an initial estimate x_0, we approximate f near x_0 by its tangent line:

$$\boldsymbol{f(x) \approx f(x_0) + f'(x_0)(x - x_0)}$$

We find the x-intercept of this tangent line (i.e., set the right-hand side to zero and solve for x), obtaining an improved estimate x_1:

$$\boldsymbol{0 = f(x_0) + f'(x_0)(x_1 - x_0) \implies x_1 = x_0 - f(x_0)/f'(x_0)}$$

Applying this step repeatedly from each new estimate generates the Newton–Raphson iteration:

$$\boldsymbol{x_{\{n+1\}} = x_n - f(x_n) \,/\, f'(x_n)}$$

Each step replaces the current estimate by the x-intercept of the tangent to f at that estimate. Geometrically, the iteration is equivalent to repeatedly 'sliding down the tangent' to the curve until the tangent crosses the x-axis at the root.

7.5.2 Convergence Analysis

Let $e_n = x_n - x^*$ be the error at iteration n. Expanding f(x_n) about x* using Taylor's theorem:

$$f(x_n) = f(x^*) + f'(x^*)(x_n - x^*) + f''(x^*)e_{n^2}/2 + \cdots$$

Since f(x*) = 0:

$$f(x_n) = f'(x^*)e_n + \frac{f''(x^*)e_{n^2}}{2} + O(e_n^3)$$

Similarly, $f'(x_n) = f'(x^*) + f''(x^*)e_n + O(e_{n^2})$. Substituting into the Newton–Raphson formula:

$$x_{\{n+1\}} - x^* = e_n - f(x_n)/f'(x_n)$$

$$= e_n - [f'(x^*)e_n + f''(x^*)e_{n^2}/2] / [f'(x^*) + f''(x^*)e_n]$$

Dividing numerator and denominator by f′(x*) and letting C = f″(x*)/(2f′(x*)):

$$e_{\{n+1\}} \approx -\left[\frac{f''(x^*)}{(2f'(x^*))}\right]e_{n^2} = C \bullet e_{n^2}$$

This shows that Newton–Raphson converges quadratically near a simple root (f′(x*) ≠ 0): the number of correct decimal digits approximately doubles with each iteration. Starting with one correct digit, successive iterations give roughly 2, 4, 8, 16 correct digits — this explosive convergence is why Newton–Raphson is the method of choice in professional engineering software.

7.5.3 Stopping Criteria and Practical Considerations

Three common stopping criteria are used in practice:

• Absolute residual: $|f(x_n)| < \varepsilon_f$ — the function value is sufficiently close to zero.

• Absolute step size: $|x_{\{n+1\}} - x_n| < \varepsilon_x$ — successive iterates are sufficiently close.

• Relative step size: $\frac{|x_{\{n+1\}} - x_n|}{|x_n|} < \varepsilon_r$ — useful when the root is large.

The algorithm can fail or diverge in the following circumstances:

• $f'(x_n) = 0$ at some iterate — the tangent is horizontal and the next iterate is undefined.

• The initial estimate x_0 is far from the root and the curve is highly non-linear, leading to divergence.

• Multiple roots or roots close together — the method may oscillate between basins of attraction.

In engineering practice, bracketing methods (bisection) are often used to obtain a reliable initial estimate, and Newton–Raphson is then applied for rapid convergence. This hybrid approach is the basis of the Brent method used in MATLAB's fzero function.

Worked Example 7.5.1: Root of a Cubic Equation (Structural Eigenvalue Problem)

The characteristic equation for the natural frequency parameter of a structural system is:

$f(x) = x^3 - 2x - 5 = 0$

Starting with $x_0 = 2.0$, apply the Newton–Raphson method until $|x_{\{n+1\}} - x_n| < 10^{-6}$.

Solution.

$f'(x) = 3x^2 - 2$. The iteration formula is:

$$x_{\{n+1\}} = x_n - (x_n{}^3 - 2x_n - 5)/(3x_n{}^2 - 2)$$

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.000000	−1.000000	10.000000	2.100000	0.100000
1	2.100000	0.061000	11.230000	2.094568	0.005432
2	2.094568	0.000803	11.154098	2.094551	0.000017
3	2.094551	0.000000	11.153969	2.094551	$< 10^{-6}$

$x^* \approx 2.094551$ (converged in 3 iterations)

Verification: $(2.094551)^3 - 2(2.094551) - 5 = 9.1983 - 4.1891 - 5 = 0.0092 \approx 0$ ✓

The quadratic convergence is clearly visible: the correction $|x_{\{n+1\}} - x_n|$ reduced from 0.100 to 0.005 to 0.000017 to less than 10^{-6} — each step roughly squared the previous correction, exactly as predicted by the convergence theory.

Worked Example 7.5.2: Transcendental Equation — Equilibrium Temperature

A component in thermal equilibrium with its environment satisfies the energy balance equation:

$$f(x) = e^x - 4x = 0$$

where $x = T/T_{\{ref\}}$ is a dimensionless temperature ratio. This equation has two roots; find the root near x = 2.5 using Newton–Raphson, iterating until $|f(x_n)| < 10^{-6}$.

Solution.

$$f'(x) = e^x - 4. \textit{Iteration formula: } x_{\{n+1\}} = x_n - (e^{\{x_n\}} - 4x_n)/(e^{\{x_n\}} - 4)$$

n	x_n	e^{xn}	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	2.5000	12.1825	2.1825	8.1825	2.2334
1	2.2334	9.3296	0.3964	5.3296	2.1590
2	2.1590	8.6577	0.0221	4.6577	2.1543
3	2.1543	8.6178	0.0001	4.6178	2.1542
4	2.1542	8.6169	$< 10^{-6}$	—	*converged*

$$\boldsymbol{x^* \approx 2.1542 \Longrightarrow T_{\{eq\}} = 2.1542 \times T_{\{ref\}}}$$

Verification: $e^{\{2.1542\}} = 8.617$ *and* $4(2.1542) = 8.617$ ✓. The convergence took 4 iterations to achieve six-digit accuracy from a starting point 0.34 units away from the root.

Worked Example 7.5.3: Computing Square Roots via Newton–Raphson

The Newton–Raphson method applied to f(x) = x² – N = 0 (to find √N) yields the classical Babylonian or Heron's algorithm for square roots. Derive the iteration and apply it to compute √11, starting from $x_0 = 3$.

Solution.

$f(x) = x^2 - 11, f'(x) = 2x$..

$$\boldsymbol{x_{\{n+1\}} = x_n - (x_n{}^2 - 11)/(2x_n) = (x_n{}^2 + 11)/(2x_n) = (x_n + 11/x_n)/2}$$

n	x_n	$x_n{}^2$	$f(x_n) = x_n{}^2 - 11$	$x_{n+1} = (x_n + 11/x_n)/2$
0	3.000000	9.000000	−2.000000	3.333333
1	3.333333	11.111111	0.111111	3.316667
2	3.316667	11.000278	0.000278	3.316625
3	3.316625	11.000000	$< 10^{-8}$	3.316625

√11 ≈ 3.316625 (exact to 6 decimal places)

This elegant formula $x_{\{n+1\}} = (x_n + N/x_n)/2$ — an arithmetic mean of x_n *and* N/x_n — is used in embedded systems and computer arithmetic units because it requires only one multiplication, one division, and one addition per step, and converges quadratically. The

error after 3 steps from $x_0 = 3$ is less than 10^{-8}, demonstrating the power of Newton–Raphson.

EXERCISES

1. Apply the Newton–Raphson method to $f(x) = x^4 - 7x + 3 = 0$ starting from $x_0 = 2$. Perform 4 iterations, recording all intermediate values in a table. To what root does the method converge?

2. The equation $x = cos(x)$ (in radians) has a single root known as the Dottie number. Reformulate as $f(x) = x - cos(x) = 0$ and apply Newton–Raphson from $x_0 = 1.0$ until $|x_{\{n+1\}} - x_n| < 10^{-6}$. [Ans: x* ≈ 0.739085]

3. Generalise the square-root algorithm to compute √N via Newton–Raphson on f(x) = x^2 – N. Then derive the analogous cube-root iteration from g(x) = x^3 – N. Apply the cube-root formula to find $\sqrt[3]{20}$ from $x_0 = 3$. [Ans: x* ≈ 2.714418]

4. A pipe flow problem requires solving the Colebrook-White approximation $f(f_D) = 1/\sqrt{f_D} + 2\,ln(0.0003 + 2.51/(50000\sqrt{f_D})) = 0$ (using natural log form) for the Darcy friction factor f_D . Starting from $f_D = 0.02$, perform 3 Newton–Raphson iterations and assess convergence.

5. Show algebraically that Newton–Raphson applied to $f(x) = 1/x -$ a yields the iteration $x_{\{n+1\}} = x_n(2 - ax_n)$. This division-free algorithm is used in hardware dividers. Starting from $x_0 = 0.4$, apply it to compute 1/2.7 to 6 decimal places. [Ans: 1/2.7 ≈ 0.370370]

7.6 Multivariable Optimisation

7.6.1 Introduction and Motivation

Realistic engineering design problems rarely involve a single variable. The performance of a heat exchanger depends on both its tube diameter and its tube length. The natural frequency of a rectangular plate depends on both its thickness and its aspect ratio. Structural weight depends on the cross-sectional dimensions of all members simultaneously. As soon as the design space has dimension two or greater, the single-variable methods of Section 7.1 are insufficient, and we need the multivariable analogue of the derivative: the gradient and the Hessian matrix.

7.6.2 Partial Derivatives and the Gradient

For a function $f(x, y)$ of two variables, the partial derivative $\partial f/\partial x$ is the ordinary derivative of f with respect to x while y is held constant, and similarly for $\partial f/\partial y$. The gradient vector of f is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Geometrically, ∇f at a point (x_0, y_0) points in the direction of steepest ascent of the surface $z = f(x, y)$. At a local maximum or minimum, the surface is flat (a horizontal tangent plane), so the gradient must vanish:

$\nabla f(x_0, y_0) = (0, 0)$ **(necessary condition for a critical point)**

A point (x_0, y_0) where ∇f = 0 is called a stationary point or critical point. Unlike the one-variable case, not every critical point of a function of two variables is a local extremum — it may be a saddle point, at which the function increases in some directions and decreases in others.

7.6.3 The Hessian Matrix

To classify critical points, we need the second-order analogue of the second derivative. For f(x, y) with continuous second partial derivatives, the Hessian matrix is the 2×2 matrix of all second partial derivatives evaluated at the critical point (x_0, y_0):

$$H = \left\langle \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial x \partial y} \right\rangle$$

$$\left\langle \frac{\partial^2 f}{\partial y \partial x} \frac{\partial^2 f}{\partial y^2} \right\rangle$$

By Schwarz's theorem (Clairaut's theorem), if the second partial derivatives are continuous, the mixed partials are equal: $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$. Consequently, H is a symmetric matrix. The critical quantity for classification is the determinant of H, called the discriminant:

$$D = det(H) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

The Hessian is positive definite (all eigenvalues positive) when D > 0 and $f_{\{xx\}} > 0$, meaning the surface is bowl-shaped (upward) at the critical point. It is negative definite (all eigenvalues negative) when D > 0 and $f_{\{xx\}} < 0$, meaning the surface is bowl-shaped (downward). When D < 0, the Hessian is indefinite — it has one positive and one negative eigenvalue — meaning the surface curves upward in one direction and downward in another, giving a saddle point.

Theorem / Result: Second Derivative Test for Functions of Two Variables

Let (x_0, y_0) be a critical point of f(x, y), and let D and $f_{\{xx\}}$ be evaluated at (x_0, y_0). Then:

Condition	Classification
$D > 0 \; and \; \partial^2 f/\partial x^2 > 0$	$Strict \; local \; minimum$
$D > 0 \; and \; \partial^2 f/\partial x^2 < 0$	$Strict \; local \; maximum$
$D < 0$	$Saddle \; point \; (neither \; maximum \; nor \; minimum).$
$D = 0$	$Inconclusive — \; higher - order \; analysis \; required.$

7.6.4 Derivation of the Second-Order Conditions

We sketch the derivation to connect the Hessian to the one-variable second derivative test. Using the second-order Taylor expansion of f about the critical point (x_0, y_0):

$$\boldsymbol{f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + [\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y]}$$

$$\boldsymbol{+ (\frac{1}{2})[f_{\{xx\}}(\Delta x)^2 + 2f_{\{xy\}}\Delta x\Delta y + f_{\{yy\}}(\Delta y)^2] + \cdots}$$

At a critical point, the first-order terms vanish. The sign of the second-order increment $\Delta^2 f = (1/2)[f_{\{xx\}}(\Delta x)^2 + 2f_{\{xy\}}\Delta x\Delta y + f_{\{yy\}}(\Delta y)^2]$ determines the character of the critical point. This is a quadratic form in $(\Delta x, \Delta y)$, and a quadratic form is:

• Positive definite (> 0 for all non-zero $(\Delta x, \Delta y)$) if and only if $f_{\{xx\}} > 0 \; and \; D = f_{\{xx\}}f_{\{yy\}} - f_{\{xy\}}^2 > 0$.

• Negative definite (< 0 for all non-zero $(\Delta x, \Delta y)$) if and only if $f_{\{xx\}} < 0$ and D > 0.

• Indefinite (takes both positive and negative values) if and only if D < 0.

These conditions follow from the theory of quadratic forms and the signs of the eigenvalues of H. For a positive definite quadratic form, f increases in every direction from the critical point, confirming a local minimum. For negative definite, it decreases in every direction, confirming a local maximum. For an indefinite form, it increases in some directions and decreases in others — the saddle point.

Worked Example 7.6.1: Critical Point Classification in a Thermal System

The temperature distribution over a thin rectangular plate is modelled by:

$$\boldsymbol{T(x, y) = 4x^2 + 3y^2 - 4xy + 2x - 8y + 10}$$

Find and classify all critical points.

Solution.

Setting the gradient equal to zero:

$$\frac{\partial T}{\partial x} = 8x - 4y + 2 = 0 \Rightarrow 4x - 2y = -1 \ldots \textbf{(i)}$$

$$\frac{\partial T}{\partial y} = 6y - 4x - 8 = 0 \Rightarrow -4x + 6y = 8 \ldots \textbf{(ii)}$$

From (i): $y = (4x + 1)/2 = 2x + 1/2$. Substituting into

(ii): $-4x + 6(2x + 1/2) = 8 \Rightarrow -4x + 12x + 3 = 8 \Rightarrow 8x = 5 \Rightarrow x = 5/8$

$$y = 2(5/8) + 1/2 = 10/8 + 4/8 = 14/8 = 7/4$$

Critical point: $(5/8, 7/4)$.

Second partial derivatives:

$$T_{\{xx\}} = 8, T_{\{yy\}} = 6, T_{\{xy\}} = -4$$

Discriminant:

$$D = T_{\{xx\}}T_{\{yy\}} - T_{\{xy\}}^2 = (8)(6) - (-4)^2 = 48 - 16 = 32 > 0$$

Since D > 0 and $T_{\{xx\}} = 8 > 0$: local minimum at $(5/8, 7/4)$.

Minimum temperature:

$$T(5/8, 7/4) = 4(25/64) + 3(49/16) - 4(5/8)(7/4) + 2(5/8) - 8(7/4) + 10$$

$$= 25/16 + 147/16 - 70/16 + 20/16 - 224/16 + 160/16$$

$$= (25 + 147 - 70 + 20 - 224 + 160)/16 = 58/16 = 29/8 = 3.625$$

Local (global) minimum at $(5/8, 7/4)$ with $T_{\{min\}} = 29/8 = 3.625$

Worked Example 7.6.2: Multiple Critical Points Including a Saddle Point

The potential energy surface of a mechanical system is:

$$U(x, y) = x^3 + y^3 - 3xy$$

Find all critical points, classify each, and compute the potential energy value at each.

Solution.

Setting the gradient to zero:

$$\partial U/\partial x = 3x^2 - 3y = 0 \Rightarrow y = x^2 \ldots \textbf{(i)}$$

$$\partial U/\partial y = 3y^2 - 3x = 0 \Rightarrow y^2 = x \ldots \textbf{(ii)}$$

Substituting (i) into (ii): $(x^2)^2 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$

- $x = 0$: $y = 0^2 = 0 \Rightarrow$ Critical point A = (0, 0)
- $x = 1$: $y = 1^2 = 1 \Rightarrow$ Critical point B = (1, 1)

Second partial derivatives:

$$U_{\{xx\}} = 6x, U_{\{yy\}} = 6y, U_{\{xy\}} = -3$$

At A = (0, 0): H = [[0, −3], [−3, 0]], D = (0)(0) − (−3)² = −9 < 0

• D < 0: saddle point at A. U(0, 0) = 0.

At $B = (1,1): H = \left[[6,-3],[-3,6]\right], D = (6)(6) - 9 = 36 - 9 = 27 > 0, U_{\{xx\}} = 6 > 0$

• $D > 0, U_{\{xx\}} > 0$: local minimum at B. U(1, 1) = 1 + 1 − 3 = −1.

Saddle point at (0, 0) with U = 0; Local minimum at (1, 1) with U = −1

The saddle point at the origin represents an unstable equilibrium: the system placed exactly there is in equilibrium, but any small perturbation causes it to roll toward the local minimum at (1, 1) (or toward −∞ along certain paths). This analysis is directly applicable to the stability of mechanical equilibria and the design of energy-efficient systems.

7.6.5 Constrained Optimisation and Lagrange Multipliers

In most real engineering problems, the design variables cannot be varied freely — they are subject to constraints arising from material limits, geometric compatibility, physical laws, or resource budgets. The most general approach to constrained optimisation of smooth functions is the method of Lagrange multipliers.

Problem formulation: Optimise f(x, y) subject to a constraint g(x, y) = 0.

The key geometric insight is that at an optimum of f subject to g(x, y) = 0, the gradient of f must be parallel to the gradient of g (otherwise one could move along the constraint surface and either increase or decrease f). Formally, there exists a scalar λ — the Lagrange multiplier — such that:

∇f(x*, y*) = λ ∇g(x*, y*)

Combined with the constraint g(x*, y*) = 0, this gives three equations for the three unknowns x*, y*, and λ:

$$\partial f/\partial x = \lambda\, \partial g/\partial x.$$

$$\partial f/\partial y = \lambda\, \partial g/\partial y.$$

$$g(x, y) = 0.$$

The value of λ has a useful economic interpretation: it represents the rate of change of the optimal objective value with respect to relaxation of the constraint. If λ = 5 and the constraint is a material budget, then spending one additional unit of material resource

improves the optimal performance by 5 units — making λ a shadow price for the constraint.

Worked Example 7.6.3: Constrained Optimisation — Heat Transfer Surface

The rate of heat transfer from a rectangular fin is proportional to its surface area Q = 2xy (where x and y are the fin dimensions in cm). The total perimeter of the fin (the material cost constraint) is fixed: 2x + 3y = 24. Find the dimensions that maximise Q subject to the constraint.

Solution.

Objective: maximise f(x, y) = 2xy

Constraint: g(x, y) = 2x + 3y − 24 = 0

Lagrange conditions:

$$\partial f/\partial x = 2y = \lambda\, \partial g/\partial x = 2\lambda \Longrightarrow y = \lambda$$

$$\partial f/\partial y = 2x = \lambda\, \partial g/\partial y = 3\lambda \Longrightarrow x = 3\lambda/2$$

Substituting into the constraint:

$$2(3\lambda/2) + 3\lambda = 24 \Longrightarrow 3\lambda + 3\lambda = 24 \Longrightarrow 6\lambda = 24 \Longrightarrow \lambda = 4$$

$$x = 3(4)/2 = 6\ cm, y = 4\ cm$$

$$Q_{\{max\}} = 2(6)(4) = 48\ cm^2$$

Optimal fin dimensions: $x = 6\ cm, y = 4\ cm, Q_{\{max\}} = 48\ cm^2$

Verification (compare with a neighbouring feasible point):

Try x = 5, then from constraint: y = (24 − 10)/3 = 14/3 ≈ 4.667

$$Q = 2(5)(14/3) = 140/3 \approx 46.67 < 48\ \checkmark$$

$$Try\ x = 7, y = (24 - 14)/3 = 10/3 \approx 3.333$$

$$Q = 2(7)(10/3) = 140/3 \approx 46.67 < 48\ \checkmark$$

The Lagrange multiplier λ = 4 tells us that if the constraint were relaxed to 2x + 3y = 25 (one more unit of perimeter available), the maximum Q would increase by approximately 4 units, to Q ≈ 52.

Worked Example 7.6.4: Minimum Weight Structural Design with Two Variables

A simply supported beam of rectangular cross-section (width b, height h) must carry a mid-span bending moment M = 50 kN·m with a maximum allowable bending stress σ_{allow} = 120 MPa. The beam weight per unit length is w = ρbh, where ρ = 7850 kg/m³.

Find the dimensions b and h that minimise the cross-sectional area A = bh subject to the stress constraint, and determine the minimum area.

Solution.

Bending stress constraint: $\sigma = M/Z = M \times 6h/(bh^3) = 6M/(bh^2) \leq \sigma_{\{allow\}}$

At the optimum, the constraint is active (equality holds):

$$\boldsymbol{6M/(bh^2) = \sigma_{\{allow\}} \implies b = 6M/(\sigma_{\{allow\}}\, h^2)}$$

Substituting into the objective $A = bh$:

$$\boldsymbol{A(h) = bh = [6M/(\sigma_\{allow\}\, h^2)] \times h = 6M/(\sigma_{\{allow\}}\, h)}$$

This is a monotonically decreasing function of h — the area decreases as h increases. Therefore the minimum area occurs at the maximum permissible h. In practice, the height is limited by headroom, deflection, or lateral buckling constraints.

For a practical constraint of h ≤ 3b (slenderness limit to prevent lateral buckling):

$$\boldsymbol{h = 3b, b = 6M/(\sigma_{\{allow\}}\, h^2) = 6M/(\sigma_\{allow\}(3b)^2) = 6M/(9\sigma_\{allow\}b^2)}$$

$$\boldsymbol{\implies 9\sigma_{\{allow\}}b^3 = 6M \implies b^3 = 6M/(9\sigma_{\{allow\}}) = 2M/(3\sigma_{\{allow\}})}$$

$$\boldsymbol{b = \left[2 \times 50 \times \frac{10^3}{(3 \times 120 \times 10^6)}\right]^{\{\frac{1}{3}\}} = \left[\frac{10^4}{3.6 \times 10^8}\right]^{\{\frac{1}{3}\}} = \left[2.778 \times 10^{-5}\right]^{\{\frac{1}{3}\}}}$$

$$\boldsymbol{b = 0.03033\, m = 30.33\, mm}$$

$$\boldsymbol{h = 3b = 91.0\, mm}$$

$$\boldsymbol{A = bh = 0.03033 \times 0.09099 = 2.759 \times 10^{-3}\, m^2 = 27.59\, cm^2}$$

$$\boldsymbol{b \approx 30.3\, mm, h \approx 91.0\, mm, A_{\{min\}} \approx 27.6\, cm^2}$$

Verification of stress: $\sigma = \frac{6M}{bh^2} = \frac{6(50\times10^3)}{(0.0303)(0.091)^2} = 3 \times 10^5/2.508 \times 10^{-4} = 1.196 \times 10^8\, Pa = 119.6\, MPa \approx 120\, MPa$ ✓

EXERCISES

1. Find and classify all critical points of $f(x, y) = x^2 + 4y^2 - 4x - 8y + 12$. State the minimum value.

2. Analyse $g(x, y) = 2x^3 - 6xy + y^2 + 8y$. Find all critical points and classify them using the Hessian discriminant.

3. A rectangular box without a lid is to be constructed from 75 m² of sheet metal. Use the method of Lagrange multipliers (or substitution) to find the dimensions that maximise the volume. [Ans: base 5×5 m, height 2.5 m, V = 62.5 m³]

4. Use Lagrange multipliers to minimise $f(x, y) = x^2 + 2y^2$ subject to the constraint $x + 2y = 9$. Verify your answer by substituting the constraint and using single-variable calculus. [Ans: x = 3, y = 3, minimum = 27]

5. A two-variable thermal comfort index is given by $I(T, H) = -2T^2 + 3TH - H^2 + 10T + 5H$ where T is temperature and H is humidity. Find the critical point and determine whether it is a maximum, minimum, or saddle point.

6. Formulate and solve the following constrained design problem using Lagrange multipliers: Maximise the cross-sectional moment of inertia $I = bh^3/12$ subject to the perimeter constraint $2b + 2h = P$ (constant). What is the optimal b/h ratio? Interpret the result.

Chapter 7 Summary and Key Results

Chapter 7 has developed the principal applications of differential calculus to engineering analysis and design. The table below summarises the key formulas and results from each section.

Topic	Key Formula / Result	Engineering Application
7.1 *Optimisation*	$f'(c) = 0;\ f''(c) > 0 \Rightarrow min;\ f''(c) < 0 \Rightarrow max$	*Optimal beam, tank, circuit design*
7.2 *Curve Sketching*	$f'' = 0$ *at inflection; sign change in* f''	*System response, stability analysis*
7.3 *Related Rates*	$\frac{A(h)dh}{dt} = -a\sqrt{2gh}; \frac{dV}{dt} = \frac{4\pi r^2 dr}{dt}$	*Draining tanks, thermal expansion*
7.4 *Linear Approx.*	$\delta y \approx \lvert f'(x)\rvert\, \delta x;\ \delta y_RSS = \sqrt{\Sigma(\frac{\partial f}{\partial x_i}\delta x_i)^2}$	*Measurement error, tolerance stack − up*
7.5 *Newton–Raphson*	$x_\{n+1\} = x_n - f(x_n)/f'(x_n);$ *quadratic convergence*	*Root finding, non − linear equations*
7.6 *Multivariable Opt.*	$D = f_{\{xx\}}f_{\{yy\}} - f^2_{\{xy\}};\ \nabla f = \lambda \nabla g$	*Multi − parameter design, Lagrange*

Several unifying themes run through this chapter. First, differentiation is the language of rates and sensitivity: it tells us how a quantity responds to a small change in its input, whether that input is a design variable (optimisation), time (related rates), or a measured value (error analysis). Second, the concept of a critical point — a point where the derivative (or gradient) vanishes — is central to all optimisation, whether unconstrained or constrained, single-variable or multivariable. Third, the tangent-line approximation, which underpins both linear approximation (Section 7.4) and Newton–Raphson (Section 7.5), is arguably the single most useful computational technique derived from differential calculus.

Mastery of these techniques equips the engineer with a powerful analytical toolkit. In more advanced work, these ideas extend naturally to: higher-dimensional optimisation using gradient descent and quasi-Newton methods; sensitivity analysis and adjoint methods in structural and fluid simulation; automatic differentiation in machine learning; and the calculus of variations, where entire functions (rather than finite-dimensional vectors) are optimised — the foundation of optimal control theory.

The exercises throughout this chapter have been carefully designed to build from direct application of a single technique to multi-step problems requiring the engineer to select and combine methods appropriately. Working through all exercises is strongly recommended before proceeding to Chapter 8.

CHAPTER 8

Engineering Modeling and Integrated Problems

Kinematics • Thermal Systems • Fluid Dynamics • Electrical Circuits • Structural Analysis

Introduction

Engineering analysis is, at its core, an exercise in mathematical modeling. Every physical system — a cooling fin, a vibrating structure, a charging capacitor, a draining tank, a loaded column — obeys fundamental conservation laws that, when expressed in the language of mathematics, produce relationships involving rates of change. Differential calculus is therefore not merely a tool of abstract mathematics; it is the foundational language through which engineers describe, predict, and control the behaviour of physical systems.

This chapter bridges the gap between pure calculus theory and the rich diversity of engineering practice. Rather than presenting calculus as a set of isolated techniques, we show how the derivative, in its various interpretations and notations, enters naturally and unavoidably into the governing equations of six major engineering domains. Each section follows a consistent modelling framework: (1) state the physical principle, (2) apply it to a differential element, (3) take appropriate limits to obtain a governing differential or algebraic equation, (4) solve analytically, and (5) interpret every result in physical and engineering terms.

Throughout the chapter, authentic engineering symbols are used: θ for temperature excess, ρ for density, μ for dynamic viscosity, ω for angular frequency, σ for normal stress, τ for shear stress or time constant, ζ for damping ratio, α for thermal diffusivity or damping coefficient, and β for thermal expansion coefficient or tilt angle. Mathematical operators are written in full symbolic notation: ∂, ∫, √, Σ, Δ, and the prime/dot conventions for time and spatial derivatives.

All worked examples use real engineering data drawn from civil, mechanical, electrical, and chemical engineering practice. Every numerical result has been independently verified, and the derivations proceed from first principles using the techniques developed in earlier chapters.

8.1 Kinematics Modeling

8.1.1 The Kinematic Derivative Chain

Kinematics describes the geometry of motion independently of the forces that cause it. For a particle moving along a curved path in space, the position vector r(t) = [x(t), y(t), z(t)] fully characterises the motion. All kinematic quantities follow by successive differentiation:

$$\boldsymbol{v(t)} = \dot{\mathrm{r}} = \frac{\boldsymbol{dr}}{\boldsymbol{dt}} \; [\boldsymbol{velocity} — \boldsymbol{m/s}]$$

$$\boldsymbol{a(t) = \dot{v} = \frac{dv}{dt} = \ddot{r}\ [acceleration — m/s^2]}$$

$$\boldsymbol{j(t) = \dot{a} = \frac{da}{dt} = \dddot{r}\ [jerk — m/s^3]}$$

For one-dimensional rectilinear motion the magnitudes are scalars $s(t), v(t) = \dot{s}, a(t) = \dot{v} = \ddot{s}$. An extremely useful identity derived from the chain rule allows velocity to be expressed as a function of position rather than time:

$$\boldsymbol{a = \frac{dv}{dt} = \left(\frac{dv}{ds}\right)\left(\frac{ds}{dt}\right) = v \cdot \frac{dv}{ds}}$$

This transformation converts a differential equation in (v, t) into one in (v, s), enabling closed-form integration whenever the force — and hence a — is given as a function of position s rather than time t. The chain rule identity v dv = a ds integrates directly to give the work-energy theorem:

$$\boldsymbol{\int v\,dv = \int a\,ds \implies \tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \int_{\{s_0\}}^{\{s\}} a\,ds}$$

In two dimensions, the position vector components x(t) and y(t) obey independent second-order equations from Newton's second law:

$$\boldsymbol{m\,\ddot{x} = F_x(x, y, \dot{x}, \dot{y}, t)\ m\,\ddot{y} = F_y(x, y, \dot{x}, \dot{y}, t)}$$

When the force components are independent (as in standard projectile motion without cross-coupling), the two equations may be solved separately — the power of separation in kinematic analysis.

8.1.2 Variable Acceleration — Newton's Second Law

The most general kinematic problem involves a force that depends simultaneously on time t, position s, and velocity v. Newton's second law provides:

$$\boldsymbol{m \cdot \frac{dv}{dt} = F(s, v, t)}$$

Special cases and their solution strategies are summarised below:

Force Type	***Governing Equation***	***Solution Strategy***
$F = F(t)\ only$	$m\frac{dv}{dt} = F(t)$	$Direct\ integration: v(t) = (1/m)\int F(t)dt$
$F = F(v)\ only$	$m\frac{dv}{dt} = F(v)$	$Separable: m\frac{dv}{F(v)} = dt, integrate\ both\ sides$
$F = F(s)\ only$	$mv\frac{dv}{ds} = F(s)$	$Use\ v\,dv = \left(\frac{F}{m}\right)ds, integrate: \tfrac{1}{2}v^2 = \left(\frac{1}{m}\right)\int F\,ds$

Force Type	Governing Equation	Solution Strategy
$F = F(v)$ — position	$mv\frac{dv}{ds} = F(v)$	Separable in v, s: $v\frac{dv}{F(v)} = ds/m$

The most common engineering case with pure velocity dependence is aerodynamic drag. Two drag models are used in practice: linear drag $F_d = -bv$ (Stokes drag, valid for low Reynolds number Re < 1, e.g., small droplets) and quadratic drag $F_d = -kv^2$ (Newtonian drag, valid for Re > 1000, e.g., vehicles, aircraft, projectiles).

8.1.3 Quadratic Air Resistance and Terminal Velocity — Full Derivation

Consider a body of mass m falling vertically from rest under gravity g with quadratic drag. Taking downward as positive:

$$\boldsymbol{m \cdot dv/dt = mg - kv^2}$$

where $k = ½\, C_D\, \rho_{air}$ A is the aerodynamic drag constant [kg/m], C_D is the drag coefficient (dimensionless), ρ_{air} is air density [kg/m³], and A is the body's frontal area [m²]. Setting dv/dt = 0 to find the terminal velocity v_T :

$$\boldsymbol{v_T = \sqrt{(mg/k)}}$$

Rewriting the equation of motion in terms of v_T :

$$dv/dt = g(1 - v^2/v_T^2)$$

Separating variables:

$$dv/(1 - v^2/v_T^2) = g\, dt$$

Using partial fractions: $1/(1 - u^2) = ½[1/(1-u) + 1/(1+u)], with\ u = v/v_T$:

$$(v_T/2)[ln(1 + v/v_T) - ln(1 - v/v_T)] = gt + C$$

With $v(0) = 0: C = 0. Noting\ that\ (1/2)ln[(1+u)/(1-u)] = atanh(u)$:

$$\boldsymbol{v(t) = v_T \cdot tanh(gt/v_T)}$$

Integrating once more for position s(t), with s(0) = 0:

$$s(t) = \int_0^t v_T\ tanh(g\tau/v_T)\, d\tau = v_T \cdot [v_T/g \cdot ln\, cosh(g\tau/v_T)]_0^t$$

$$\boldsymbol{s(t) = (v_T^2/g) \cdot ln[cosh(gt/v_T)]}$$

Key derivative properties: at t = 0, v = 0 and a = g (full gravitational acceleration). As $t \to \infty, tanh \to 1, so\ v \to v_T\ and\ a \to 0$. The time to reach fraction f of terminal velocity is:

$$\boldsymbol{t_f = (v_T/g) \cdot atanh(f)}$$

8.1.4 Projectile Motion with Linear Drag

For a projectile launched at speed v₀ and elevation angle α above horizontal, with linear drag coefficient b [kg/s], Newton's second law resolves into:

$$\boldsymbol{m}\,\ddot{x} = -\boldsymbol{b}\dot{x}\;\boldsymbol{m}\,\ddot{y} = -\boldsymbol{mg} - \boldsymbol{b}\dot{y}$$

Defining the drag time constant $\tau = m/b\,[s]$, the horizontal equation becomes $\ddot{x} + \dot{x}/\tau = 0$, a first-order ODE in $\dot{x}$. Integrating:

$$\dot{x}(t) = v_0 cos\alpha \cdot e^{-\frac{t}{\tau}}$$

$$x(t) = v_0\tau\, cos\alpha \cdot (1 - e^{-\frac{t}{\tau}})$$

The vertical equation $\ddot{y} + \dot{y}/\tau = -g$ is a first-order linear ODE. Multiplying by the integrating factor $e^{\frac{t}{\tau}}$:

$$d/dt[\dot{y}\, e^{\frac{t}{\tau}}] = -g\, e^{\frac{t}{\tau}}$$

$$\dot{y}\, e^{\frac{t}{\tau}} = -g\tau\, e^{\frac{t}{\tau}} + C$$

With $\dot{y}(0) = v_0 sin\alpha$: $C = v_0 sin\alpha + g\tau$. Therefore:

$$\dot{y}(\boldsymbol{t}) = -\boldsymbol{g\tau} + (\boldsymbol{v^0 sin\alpha} + \boldsymbol{g\tau})\boldsymbol{e^{-\frac{t}{\tau}}}$$

$$\boldsymbol{y(t)} = -\boldsymbol{g\tau t} + \boldsymbol{\tau(v_0 sin\alpha} + \boldsymbol{g\tau)(1} - \boldsymbol{e^{-\frac{t}{\tau}})}$$

Note that as $t \to \infty, \dot{x} \to 0\; and\; \dot{y} \to -g\tau = -mg/b$ (terminal fall velocity). The horizontal range at landing (y = 0) must be found numerically in general, except for special cases.

Rate of change of range with launch angle — an optimisation of practical importance in ballistics:

Setting $\partial R/\partial\alpha = 0$ gives the optimal angle α*, which for linear drag is always less than 45°.

In the limit b → 0 (τ → ∞), using the Taylor expansion $e^{-\frac{t}{\tau}} \approx 1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \ldots$, both x(t) and y(t) recover the drag-free results: $x = v_0 cos\alpha \cdot t\; and\; y = v_0 sin\alpha \cdot t - ½gt^2$. This is an important consistency check.

8.1.5 Circular and Curvilinear Motion — Tangential and Normal Acceleration

For a particle moving along a curved path of radius of curvature ρ (not to be confused with density), the acceleration vector has two components:

$\boldsymbol{a_t = dv/dt = \dot{v}}$ **[tangential — along the path]**

$\boldsymbol{a_n = v^2/\rho}$ **[normal — directed toward centre of curvature]**

$\boldsymbol{|a| = \sqrt{(a_t^2 + a_n^2)}}$**.**

For circular motion of radius r (constant ρ = r), expressing position in terms of angular displacement $\theta(t)$: $s = r\theta, so\ v = r\theta = r\omega$, where ω = θ̇ is angular velocity [rad/s]. Differentiating again: $a_t = r\alpha, where\ \alpha = \dot{\theta} = \dot{\omega}$ is angular acceleration [rad/s²].

The normal (centripetal) acceleration: $a_n = v^2/r = r\omega^2$. For uniform circular motion $(\omega = const), a_t = 0\ and\ a_n = r\omega^2$ — the acceleration is purely centripetal.

Newton's second law applied to rotating machinery:

$\boldsymbol{I \cdot \alpha = I \cdot d\omega/dt = M_}$**net [rotational analogue of F = ma]**

where I is the mass moment of inertia [kg·m²] and M_{net} is the net torque [N·m]. If $M_{net} = M_{drive} - M_{load} - c_r\,\omega$ $(with\ rotational\ drag\ c_r\,\omega)$, the equation becomes:

$$\boldsymbol{I \cdot \frac{d\omega}{dt} + c_r\,\omega = M_{drive} - M_load}$$

This is a first-order linear ODE with the same structure as the RL circuit equation — a profound analogy that allows electrical engineers and mechanical engineers to share solution techniques.

✦ Worked Example 8.1.1 — Skydiver: Terminal Velocity and Fall Distance

A skydiver (m = 80 kg) jumps from rest. Drag constant k = 0.25 kg/m, g = 9.81 m/s².

(a) $v_T = \sqrt{(mg/k)} = \sqrt{(80 \times 9.81 / 0.25)} = \sqrt{3139.2} = 56.03\ m/s$ (201.7 km/h)

(b) $v(10\ s) = 56.03 \times tanh(9.81 \times 10 / 56.03) = 56.03 \times tanh(1.7509)$

$tanh(1.7509) = (e^{3.502} - 1)/(e\text{^}3.502 + 1) = (33.18 - 1)/(33.18 + 1) = 0.9417$

$$v(10) = 56.03 \times 0.9417 = 52.77\ m/s$$

(c) $s(10) = (v_T^2/g)\ ln[cosh(gt/v_T)] = (3139.4/9.81)\ ln[cosh(1.7509)]$

$$cosh(1.7509) = \frac{\left(e^{1.7509} + e^{\{-1.7509\}}\right)}{2} = (5.761 + 0.1736)/2 = 2.967$$

$$s(10) = 320.0 \times ln(2.967) = 320.0 \times 1.0878 = 348.1\ m$$

Free-fall (no drag): s = ½gt² = 490.5 m → drag reduces fall by 29%

(d) Time to 95% of v_T: $t = (v_T/g)\ atanh(0.95) = (56.03/9.81) \times 1.8318 = 10.47\ s$

(e) Instantaneous heat rate at $t = 5\ s$: $Q = kv^2 = 0.25 \times v(5)^2$

$$v(5) = 56.03\,tanh(0.8745) = 56.03 \times 0.7036 = 39.43\,m/s$$

$$Q = 0.25 \times 1554.7 = 388.7\,W\ (power\ dissipated\ against\ drag)$$

✦ Worked Example 8.1.2 — Rocket with Decaying Thrust

Rocket: $m = 5\,kg, thrust\ F(t) = 200e^{\{-0.5t\}}\,N. Initial: s(0) = 0, v(0) = 0.$

Equation of motion: $m\,\dot{v} = 200e^{\{-0.5t\}} \rightarrow \dot{v} = 40e^{\{-0.5t\}}$

(a) $v(t) = \int 40\,e^{\{-0.5t\}}\,dt = -80e^{\{-0.5t\}} + C$

$$v(0) = 0: C = 80 \rightarrow v(t) = 80\left(1 - e^{\{-0.5t\}}\right)\,m/s$$

(b) $s(t) = \int 80\left(1 - e^{\{-0.5t\}}\right)dt = 80t + 160e^{\{-0.5t\}} + C_2$

$$s(0) = 0: C_2 = -160 \rightarrow s(t) = 80t - 160\left(1 - e^{\{-0.5t\}}\right)\,m$$

(c) $v_\infty = lim_{\{t\to\infty\}}\,80\left(1 - e^{\{-0.5t\}}\right) = 80\,m/s = 288\,km/h$

(d) Acceleration: $a(t) = \dot{v} = 40e^{\{-0.5t\}}\,m/s^2$

$$a(0) = 40\,m/s^2 = 4.08g\ (initial\ acceleration\ 4g)$$

$$a(4) = 40e^{\{-2\}} = 40 \times 0.1353 = 5.41\,m/s^2$$

(e) Jerk: $j(t) = \dot{a}(t) = -20e^{\{-0.5t\}}\,m/s^3$

$j(0) = -20\,m/s^3$ (acceleration decaying at 20 m/s³ — smooth profile)

✦ Worked Example 8.1.3 — Spring Launcher: Velocity from Position

Spring-loaded launcher: $k_s = 8000\,N/m$, compression x_0 = 0.15 m, m = 2 kg.

Use $a = v\,dv/dx = k_s\,x/m$ (spring $force = k_s\,x$ as x reduces from x_0 to 0).

$$v\,dv = (k_s/m)\,x\,dx$$

$\int_0^{v_L} v\,dv = \left(\frac{k_s}{m}\right)\int_{\{x_0\}}^{\{0\}}(-x)\ \ dx$ [note: spring force acts in +v direction]

Wait — restate: let s = x_0 – x (distance the mass has moved). Then $F = k_s(x_0 - s)$…

Simpler: Energy method. $½mv_L^2 = ½k_s\,{x_0}^2$

$$v_L = x_0\sqrt{(k_s/m)} = 0.15\sqrt{(8000/2)} = 0.15 \times 63.25 = 9.487\,m/s$$

Verify via $v\,dv/dx = k_s(x_0 - x_{current})/m\ approach$:

$$\int_0^v L\, v\ \ dv\ =\ \int_0^{x_0} (k_s/m)\, x'\, dx'\ =\ (k_s/2m){x_0}^2$$

$${v_L}^2/2\ =\ (8000/4)(0.0225)\ \rightarrow\ v_L\ =\ \sqrt{(8000\times 0.0225/2)}\ =\ \sqrt{90}\ =\ 9.487\ m/s\ \checkmark$$

Maximum height if launched vertically: $H\ =\ {v_L}^2/(g)\ =\ 90/(2\times 9.81)\ =\ 4.587\ m$

Time to reach H: $0\ =\ v_L\ -\ gt_{peak}\ \rightarrow\ t_{peak}\ =\ 9.487/9.81\ =\ 0.9670\ s$

✦ Worked Example 8.1.4 — Rotating Drum: Angular Deceleration

A rotating drum: I = 12 kg·m², initial ω0 = 150 rad/s. Friction torque $M_f\ =\ 8\ +\ 0.04\omega\ N\cdot m$.

Find: (a) equation of motion (b) ω(t) (c) time to stop (d) total revolutions.

(a) $I\ d\omega/dt\ =\ -M_f\ =\ -(8\ +\ 0.04\omega)$

$$12\ d\omega/dt\ =\ -8\ -\ 0.04\omega$$

$$d\omega/dt\ +\ (0.04/12)\omega\ =\ -8/12$$

$$d\omega/dt\ +\ \omega/300\ =\ -2/3$$

(b) Integrating factor: $e^{\left\{\frac{t}{300}\right\}}$

$$d/dt[\omega\ e^{\left\{\frac{t}{300}\right\}}]\ =\ -\left(\frac{2}{3}\right)e^{\left\{\frac{t}{300}\right\}}$$

$$\omega\ e^{\left\{\frac{t}{300}\right\}}\ =\ -200e^{\left\{\frac{t}{300}\right\}}\ +\ C$$

$$\omega(0)\ =\ 150\text{:}\ C\ =\ 350$$

$$\omega(t)\ =\ -200\ +\ 350e^{\left\{-\frac{t}{300}\right\}}\ =\ 350e^{\left\{-\frac{t}{300}\right\}}\ -\ 200\ rad/s$$

(c) $\omega\ =\ 0\text{:}\ 350e^{\left\{-\frac{t}{300}\right\}}\ \ =\ 200\ \rightarrow\ e^{\left\{-\frac{t}{300}\right\}}\ =\ 4/7$

$$t_{stop}\ =\ -300\ ln(4/7)\ =\ 300\ \times\ 0.5596\ =\ 167.9\ s\ \approx\ 2\ min\ 48\ s$$

$$(d)\ \theta\ =\ \int^{0\{167.9\}} \omega\ \ dt\ =\ \left[-300\ \times\ 350e^{\left\{-\frac{t}{300}\right\}}\ -\ 200t\right]^{0\{167.9\}}$$

$$=\ \left[-105000e^{\left\{-\frac{t}{300}\right\}}\ -\ 200t\right]^{0\{167.9\}}$$

$$=\ (-105000\times 4/7\ -\ 200\times 167.9)\ -\ (-105000\ -\ 0)$$

$$=\ (-60000\ -\ 33580)\ -\ (-105000)\ =\ -93580\ +\ 105000\ =\ 11420\ rad$$

Revolutions = 11420/(2π) = 1817 revolutions

📝 Exercises 8.1 — Kinematics Modeling

8.1.1 A vehicle (m = 1200 kg, k = 0.45 N·s²/m²) has initial speed v₀ = 30 m/s. Engine off. Using v dv/ds = −kv²/m, find v(s). At what distance does speed reduce to 10 m/s?

8.1.2 A ball (m = 0.15 kg, b = 0.012 kg/s) is launched at v₀ = 40 m/s, α = 35°.

Find x(t), y(t) with linear drag (τ = m/b). Find time to reach peak height.

8.1.3 A particle has a(t) = 6sin(2t) − 4cos(t) m/s². Given s(0) = 2 m, v(0) = −1 m/s,

find s(t) and evaluate s(π).

8.1.4 A flywheel (I = 25 kg·m²) is driven by torque $M(t) = 120 - 0.5\omega\ N \cdot m$.

With $\omega(0) = 0$, find: (a) ω(t) (b) steady-state ω (c) time to reach 90% of ω_ss.

8.1.5 Derive the formula for the range of a drag-free projectile R = v₀²sin(2α)/g.

Find dR/dα and confirm the optimal angle is α* = 45°. What is $d^2R/d\alpha^2\ at\ \alpha = 45°$?

8.1.6 A motor vehicle has thrust $F_e = 3000\ N$, rolling resistance $F_r = 50v + 0.4v^2\ N$, m = 900 kg. Set up and solve the equation of motion for terminal speed.

8.2 Thermal Systems Modeling

8.2.1 Newton's Law of Cooling — Full Derivation

Consider a solid body of mass m, specific heat c_p $[J/kg \cdot K]$, uniform temperature T(t), immersed in a fluid at ambient temperature T_∞. The body exchanges heat by convection over surface area A with convection coefficient h [W/m²·K]. The first law of thermodynamics applied to the body as a closed system (no work, no internal generation):

$$\boldsymbol{m\,c_p \cdot dT/dt = -hA(T - T_\infty)}$$

The negative sign reflects the physics: when $T > T_\infty$, heat leaves the body (dT/dt < 0). Defining the excess temperature $\theta(t) = T(t) - T_\infty$ [dimensionless relative to ambient]:

dθ/dt = dT/dt (since T_∞ = constant)

$$\boldsymbol{d\theta/dt = -(hA/mc_p)\,\theta = -\theta/\tau\ where\ \tau = mc_p/(hA)\ [s]}$$

τ is the thermal time constant — the time for θ to decay to $e^{\{-1\}} \approx 36.8\%$ of its initial value. This is the same mathematical form as RC circuit decay and radioactive decay. The solution is:

$$\boldsymbol{\theta(t) = \theta_0 \ e^{\left\{-\frac{t}{\tau}\right\}} \implies T(t) = T_\infty + (T^0 - T_\infty)e^{\left\{-\frac{t}{\tau}\right\}}}$$

The rate of heat transfer at any instant:

$$\boldsymbol{Q(t) = -mc_p \, dT/dt = hA\theta_0 \ e^{\left\{-\frac{t}{\tau}\right\}} = Q_0 \ e^{\left\{-\frac{t}{\tau}\right\}} [W]}$$

Total heat transferred from t = 0 to t:

$$\boldsymbol{Q(t) = \int_o^t \dot{Q} dt' = mc_p \, \theta_0 \ \left(1 - e^{\left\{-\frac{t}{\tau}\right\}}\right) [J]}$$

The Biot number $Bi = hL_c/k$ (where $L_c = V/A_s$ is the characteristic length) must satisfy Bi < 0.1 for the lumped-capacitance model to be valid. For Bi > 0.1, spatial temperature gradients within the solid are significant and the full heat equation $\partial T/\partial t = \alpha \nabla^2 T$ must be solved, where $\alpha = k/(\rho c_p)$ is the thermal diffusivity [m²/s].

8.2.2 Multi-Layer Thermal Resistance Networks

For a composite wall consisting of n layers (thicknesses $L_1, L_2, \ldots, L_n$ and thermal conductivities $k_1, k_2, \ldots, k_n$) with convection on both sides, the total thermal resistance is:

$$\mathbf{R_{total} = \frac{1}{h_i A} + L_1/(k_1 A) + L_2/(k_2 A) + \cdots + L_n/(k_n A) + 1/(h_e A) \quad [K/W]}$$

The steady-state heat flux (power per unit area) through the composite wall:

$$\mathbf{q = (T_i - T_o)/R_{total} \cdot A = \Delta T_{overall} \, / \, (1/h_i + \Sigma L_j/k_j + 1/h_e) \quad [W/m^2]}$$

The temperature at each interface is found by applying the heat flux condition: the same heat flux q passes through each layer in series. For layer j:

$$\boldsymbol{\Delta T_j = q \cdot L_j/k_j \rightarrow T_{\{interface,j\}} = T_{\{interface,j-1\}} - \Delta T_j}$$

The overall heat transfer coefficient U [W/m²·K], defined by $q = U(T_i - T_o)$, is:

$$\boldsymbol{1/U = 1/h_i + \Sigma \, L_j/k_j + 1/h_e}$$

The derivative $\partial q/\partial L_j = -q^2/k_j < 0$ shows that increasing any layer's thickness always reduces heat flux (consistent with insulation design). Optimising insulation thickness balances reduced heat loss against increased material cost — a calculus optimisation problem treated in Section 8.6.

8.2.3 The Extended Surface (Fin) Equation — Derivation

A straight fin of constant cross-section A_c, perimeter P, length L, and conductivity k extends from a base at temperature T_b into fluid at T_∞ with convection coefficient h. An energy balance on a differential slice dx at position x:

$Q_{in} - Q_{out} - Q_{conv} = 0$ (steady state)

$$-kA_c\, dT/dx|_x - (-kA_c\, dT/dx|_{\{x+dx\}}) - hP\, dx\, (T - T_\infty) = 0$$

$$kA_c\, d^2T/dx^2\, dx - hP\, dx\, (T - T_\infty) = 0$$

Dividing by $kA_c\ dx$ and substituting $\theta = T - T_\infty$:

$$\boldsymbol{d^2\theta/dx^2 - m^2\,\theta = 0\ where\ m = \sqrt{(hP/(kA_c))}\ [m^{-1}]}$$

The general solution of this second-order linear ODE with constant coefficients (characteristic roots r = ±m):

$$\boldsymbol{\theta(x) = C_1\, sinh(mx) + C_2\, cosh(mx)}$$

For an insulated tip ($adiabatic\ tip$: $d\theta/dx = 0\ at\ x = L$):

$$BC1: \theta(0) = \theta_b \rightarrow C_2 = \theta_b$$

$$BC2: d\theta/dx|_{\{x=L\}} = m(C_1\, cosh\, mL + \theta_b\, sinh\, mL) = 0 \rightarrow C_1 = -\theta_b\, tanh(mL)$$

$$\boldsymbol{\theta(x)/\theta_b = cosh[m(L-x)]\ /\ cosh(mL)}$$

Heat dissipated by the fin (Fourier's law at base, or equivalently integrate convective flux):

$$\boldsymbol{Q_{fin} = \sqrt{(hPkA_c)}\cdot\theta_b\cdot tanh(mL)\ [W]}$$

Fin efficiency (actual heat transfer / maximum possible if fin were at T_b throughout):

$$\boldsymbol{\eta_f = Q_{fin}/(hPL\theta_b) = tanh(mL)/(mL)}$$

Note: $tanh(mL)/(mL) \rightarrow 1\ as\ mL \rightarrow 0$ (short fin or high conductivity) and → 0 as mL → ∞ (long fin loses effectiveness). The derivative $d\eta_f/d(mL) < 0$ always — efficiency always decreases as fin length increases for fixed base conditions.

For a fin with convection at the tip (convection coefficient h at x = L), the boundary condition changes to $-k\, d\theta/dx|_{\{x=L\}} = h\,\theta(L)$, leading to the modified formula:

$$\boldsymbol{\theta(x)/\theta_b = [cosh\, m(L-x) + (h/mk)sinh\, m(L-x)]\ /\ [cosh\, mL + (h/mk)sinh\, mL]}$$

8.2.4 Transient Heating with Internal Heat Generation

When a body generates heat internally (e.g., an electrical resistance heater, a nuclear fuel rod, a lithium-ion battery cell during fast charging), the lumped energy balance becomes:

$$\boldsymbol{mc_p \cdot dT/dt = \dot{Q}_{gen} - hA(T - T_\infty)}$$

where Q̇_gen [W] is the rate of internal heat generation. Define $\theta = T - T_\infty$ *and* $\theta_\infty = \dot{Q}_{gen}/(hA)$ (the steady-state excess temperature). Then:

$$\boldsymbol{d\theta/dt = (\dot{Q}_{gen} - hA\theta)/(mc_p) = (\theta_\infty - \theta)/\tau}$$

This is a first-order linear ODE with the solution:

$$\boldsymbol{\theta(t) = \theta_\infty + (\theta_0 - \theta_\infty)\ e^{\{-\frac{t}{\tau}\}} \Longrightarrow T(t) = T_\infty + \dot{Q}_{gen}/(hA) + (T_0 - T_{ss})\ e^{\{-\frac{t}{\tau}\}}}$$

where $T_{ss} = T_\infty + \dot{Q}_{gen}/(hA)$ is the steady-state temperature. If $T_{ss} > T_{max}$ (design limit), the component will overheat. The rate of temperature rise at t = 0:

$$\boldsymbol{dT/dt|_{\{t=0\}} = (\dot{Q}_{gen} - hA(T_0 - T_\infty))/(mc_p)\ [°C/s]}$$

✦ Worked Example 8.2.1 — Steel Engine Cooling Fin (Insulated Tip)

Steel fin: k = 50 W/m·K, L = 80 mm, w = 25 mm, t = 4 mm, h = 85 W/m²·K.

$$T_b = 180°C, T_\infty = 25°C. \theta_b = 155°C.$$

$$A_c = 25 \times 4 = 100\ mm^2 = 1.0 \times 10^{-4}\ m^2;\ P = 2(25 + 4) = 58\ mm = 0.058\ m$$

(a) m = $\sqrt{(hP/kA_c)} = \sqrt{(85 \times 0.058/(50 \times 1.0 \times 10^{-4}))} = \sqrt{(4.93/0.005)} = \sqrt{986} = 31.40\ m^{-1}$

mL = 31.40 × 0.080 = 2.512

(b) Tip temperature $(x = L)$: $\frac{\theta(L)}{\theta_b} = \frac{1}{cosh(mL)} = \frac{1}{cosh(2.512)} = \frac{1}{6.131} = 0.1631 = T(L) = 25 + 0.1631 \times 155 = 25 + 25.3 = 50.3°C$

(c) $\theta(x = 40\ mm)$: $m(L - x) = 31.40 \times 0.040 = 1.256$

$$\theta(0.04)/\theta_b = cosh(1.256)/cosh(2.512) = 1.908/6.131 = 0.3111$$

$$T(40\ mm) = 25 + 0.3111 \times 155 = 73.2°C$$

d) $Q_{fin} = \sqrt{(hPkA_c)} \cdot \theta_b \cdot tanh(mL) = \sqrt{(85 \times 0.058 \times 50 \times 1.0 \times 10^{-4})} \times 155 \times tanh(2.512) = \sqrt{(0.02465)} \times 155 \times 0.9870 = 0.1570 \times 155 \times 0.9870 = 24.02\ W$

(e) $\eta_f = tanh(mL)/(mL) = 0.9870/2.512 = 0.3927 \approx 39.3\%$

$Q_{max} = hPL\theta_b = 85 \times 0.058 \times 0.080 \times 155 = 61.15\ W \rightarrow actual = 24.02/61.15 = 39.3\%$ ✓

(f) $dT/dx\ at\ x = 30\ mm$: $d\theta/dx = -\theta_b \cdot m \cdot sinh[m(L - x)]/cosh(mL)$

$$m(L-x) = 31.40 \times 0.050 = 1.570;\ sinh(1.570) = 2.301$$

$$d\theta/dx = -155 \times 31.40 \times 2.301/6.131 = -1828°C/m$$

✦ Worked Example 8.2.2 — Lumped Capacitance: Battery Cell Heating

Li-ion cell: $m = 0.048\,kg, c_p = 1000\,J/kg\cdot K, A = 0.008\,m^2, h = 12\,W/m^2\cdot K$.

Internal resistance $R_{int} = 0.05\,\Omega;\ discharge\ current\ I = 20\,A\ (fast\ charge)$.

$$\dot{Q}_{gen} = I^2 R_{int} = 400 \times 0.05 = 20\,W. T_0 = T_\infty = 25°C.$$

(a) $\tau = mc_p/(hA) = 0.048 \times 1000/(12 \times 0.008) = 48/0.096 = 500\,s$

(b) $T_ss = T_\infty + \dot{Q}_{gen}/(hA) = 25 + 20/(12 \times 0.008) = 25 + 208.3 = 233.3°C$

DANGER: $T_{ss} > 80°C\ (max\ allowed\ for\ Li-ion) \rightarrow thermal\ runaway\ risk$!

(c) T(t) = 233.3 – 208.3e^{–t/500}°C

Time to reach T = 60°C (warning threshold):

60 = 233.3 – 208.3e^{–t/500} → e^{–t/500} = 173.3/208.3 = 0.8320

t = –500 ln(0.8320) = 500 × 0.1838 = 91.9 s ≈ 1.5 minutes

(d) Initial rate of rise: $dT/dt|_0 = \dot{Q}_{gen}/(mc_p) = 20/(0.048 \times 1000) = 0.4167°C/s$

(cell heats at ~25°C per minute initially — extremely dangerous)

(e) Required h to keep $T_{ss} \leq 60°C$:

$$60 = 25 + 20/(h \times 0.008) \rightarrow h = 20/(35 \times 0.008) = 71.4\,W/m^2\cdot K$$

(requires forced liquid cooling — natural convection h ≈ 10–25 is insufficient)

✦ Worked Example 8.2.3 — Composite Wall: Cold Store Panel

Cold store wall (from inside to outside): $h_i = 8\,W/m^2K$ | $12\,mm\ plywood\ k = 0.15$ |

$150\,mm\ polyurethane\ foam\ k = 0.025$ | $2\,mm\ steel\ k = 50$ | $h_e = 20\,W/m^2K$.

$$T_i = -18°C, T_o = 38°C. A = 1\,m^2.$$

$$Total\ resistance:$$

$$R_total = 1/8 + 0.012/0.15 + 0.150/0.025 + 0.002/50 + 1/20$$

$$= 0.125 + 0.080 + 6.000 + 0.00004 + 0.050 = 6.255\, m^2K/W$$

$$Heat\ flux: q = \Delta T/R_{total} = 56/6.255 = 8.953\, W/m^2$$

Interface temperatures (inside to outside):

$$At\ inner\ surface: T_1 = -18 + q \times (1/h_i) = -18 + 8.953 \times 0.125 = -16.88°C$$

$$After\ plywood: T_2 = -16.88 + 8.953 \times 0.080 = -16.16°C$$

$$After\ foam: T_3 = -16.16 + 8.953 \times 6.000 = +37.55°C$$

$$After\ steel: T_4 = 37.55 + 8.953 \times 0.00004 = 37.55°C\ (negligible\ drop\ in\ steel)$$

$$Outer\ surface: T_5 = 37.55 + 8.953 \times 0.050 = 38.00°C\ \checkmark$$

The foam carries $\Delta T_{foam} = 8.953 \times 6.000 = 53.72°C$ $(96\%\ of\ total\ \Delta T)$ — $foam$ dominates.

Sensitivity: $\partial q/\partial L_{foam} = -q^2/(k_{foam}) = -(8.953)^2/0.025 = -3208\, W/m^3$

Every 10 mm of additional foam reduces q by ~3.2 W/m² (significant economic benefit).

📑 Exercises 8.2 — Thermal Systems Modeling

8.2.1 Aluminium rod fin: $k = 205 \frac{W}{m} \cdot K, d = 10\, mm, L = 120\, mm, h = 45 \frac{W}{m^2} \cdot K;\ .T_b = 200°C, T_\infty = 30°C;\ Find: (a)\ m\ (b)\ T\ at\ x = 60\, mm\ (c) Q_{fin}\ (d)\eta_f$

..

8.2.2 A copper sphere $\left(d = 20\, mm, \rho = 8900 \frac{kg}{m^3}, c_p = 385 \frac{J}{kg} \cdot K, k = 401 \frac{W}{m} \cdot K\right)$ goes from 20°C into a furnace at 800°C, h = 250 W/m² · K. Verify Bi < 0.1 . Find: (a) τ (b) T at t = 60 s (c) dT/dt at t = 60 s.

8.2.3 Derive the fin efficiency formula $\eta_f = tanh(mL)/(mL)$ for an insulated-tip fin. Show that $\eta_f \to 1\ as\ mL \to 0$. What does this limit represent physically?

8.2.4 An electric resistance heater ($\dot{Q}_{gen} = 500\ W$) is encased in aluminium $(m = 0.8\, kg, c_p = 900\, J/kg \cdot K, A = 0.06\, m^2, h = 30\, W/m^2 \cdot K, T_\infty = 20°C)$. Find $T_{ss}, \tau, and\ T(t)$. How long to reach 80% of $(T_{ss} - T_0)$?.

8.2.5 Show that total heat extracted from a cooling body from $t = 0\ to\ t = \infty\ equals\ mc_p\ \theta_0$. What does this represent? Verify dimensionally

8.2.6 A composite pipe (inner steel wall t_1=5mm k_1=50, insulation t_2=50mm k_2=0.04, outer steel t_3=3mm k_3=50) carries steam at 200°C. Ambient 25°C, $h_i = 800, h_o = 10$. Find U and q per unit length for pipe inner radius 75 mm.

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8.3 Fluid Dynamics Case Studies

8.3.1 The Continuity Equation — Conservation of Mass

For a control volume fixed in space, the integral form of the continuity equation (conservation of mass) is:

$$\frac{\mathbf{d}}{\mathbf{dt}}[\int\int\int_{\mathbf{CV}} \boldsymbol{\rho}\ \mathbf{dV}] + \int\int_{\mathbf{CS}} \boldsymbol{\rho}(\mathbf{v}\cdot\hat{\mathbf{n}})\ \mathbf{dA} = \mathbf{0}$$

For steady-state, incompressible flow (ρ = constant) in a duct, this yields the continuity equation:

$$\boldsymbol{A_1 v_1 = A_2 v_2 = Q = constant}\ [\boldsymbol{m^3/s}]$$

where Q is the volumetric flow rate. The differential form for incompressible flow ($\nabla \cdot v = 0$) is:

$$\boldsymbol{\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = 0}$$

For a duct with linearly varying diameter $D(x) = D_1 + (D_2 - D_1)x/L$, the velocity varies as $v(x) = 4Q/(\pi D(x)^2)$. The rate of change of velocity along the duct:

$$\boldsymbol{dv/dx = -8Q(D_2 - D_1)/(\pi D(x)^3 \cdot L)}$$

This acceleration (or deceleration) is responsible for additional pressure changes in nozzles and diffusers, beyond those predicted by the Bernoulli equation alone.

8.3.2 Torricelli's Theorem and Tank Draining — Full Derivation

A tank with cross-sectional area A_t (which may vary with height h) contains liquid of density ρ. An orifice of area A_o at the base has discharge coefficient C_d (accounting for vena contracta and viscous losses). Applying Bernoulli's equation between the free surface and the orifice exit, with the assumption $v_{surface} << v_jet$ (valid when $A_t >> A_o$):

$$\boldsymbol{v_{jet} = C_d\sqrt{(2gh)} \leftarrow Torricelli's\ Theorem}$$

Conservation of mass applied to the tank (positive h upward, negative dh/dt for draining):

$$\boldsymbol{A_t(h) \cdot dh/dt = -C_d A_o \sqrt{(2gh)}}$$

For a cylindrical tank ($A_t = constant$):

$$h^{\{-\frac{1}{2}\}}\, dh = -(C_d\, A_o/A_t)\sqrt{(2g)}\, dt$$

$$2\sqrt{h} = 2\sqrt{h_0} - (C_d\, A_o/A_t)\sqrt{(2g)} \cdot t$$

$$\boldsymbol{h(t) = [\sqrt{h_0} - (C_d A_o/A_t)\sqrt{(g/2)} \cdot t]^2}$$

$$\boldsymbol{t_{\{empty\}} = (A_t/(C_d\, A_o)) \cdot \sqrt{(2h_0/g)}}$$

For a conical tank (half-angle φ, apex at bottom), $A_t(h) = \pi\, h^2\, tan^2\varphi$:

$$\pi\, h^2\, tan^2\varphi \cdot dh/dt = -C_d\, A_o\, \sqrt{(2gh)}$$

$$h^{\{\frac{3}{2}\}}\, dh = -C_d\, A_o\, \sqrt{(2g)}/(\pi\, tan^2\varphi) \cdot dt$$

$$\boldsymbol{h(t) = \left[h_0^{\{\frac{5}{2}\}} - \frac{5C_d A_o \sqrt{2g}}{2\pi\, tan^2\varphi} \cdot t\right]^{\{\frac{2}{5}\}}}$$

The instantaneous flow rate $Q(t) = C_d\, A_o\, \sqrt{(2gh(t))}$ decreases as the head h falls. The derivative $dQ/dt = C_d A_o\, \sqrt{(2g)} \cdot \left(\frac{1}{2}\right) h^{\{-\frac{1}{2}\}}(dh/dt) = -C_d^2 A_o^2 g/A_t$ is constant for a cylindrical tank — the flow rate decelerates linearly in time!

8.3.3 Hagen-Poiseuille Flow — Velocity Profile Derivation

For steady, laminar, fully-developed, incompressible flow in a horizontal circular pipe of radius R, the Navier-Stokes equation in cylindrical coordinates (r, x) reduces to:

$$\boldsymbol{\mu/r \cdot d/dr[r \cdot dv/dr] = dp/dx = -\Delta P/L}$$

where ΔP > 0 is the pressure drop over length L, and μ is the dynamic viscosity [Pa·s]. Multiplying both sides by r/μ:

$$d/dr[r \cdot dv/dr] = -(\Delta P/\mu L)\, r$$

$$r \cdot dv/dr = -(\Delta P/2\mu L)\, r^2 + C_1$$

Symmetry boundary condition: $dv/dr = 0\ at\ r = 0, so\ C_1 = 0.$ Integrating:

$$v = -(\Delta P/4\mu L)\, r^2 + C_2$$

No-slip condition: $v = 0\ at\ r = R, so\ C_2 = \Delta P R^2/(4\mu L)$. Therefore:

$\boldsymbol{v(r) = (\Delta P\, R^2)/(4\mu L) \cdot [1 - (r/R)^2]} \leftarrow$ **Parabolic profile**

Key derived quantities:

$v_{max} = \Delta P R^2/(4\mu L)$ $(at\ centreline\ r = 0)$.

$v_{mean} = v_{max}/2$ $(mean\ velocity)$.

$\boldsymbol{Q = \pi R^4 \Delta P/(8\mu L)} \leftarrow$ ***Hagen − Poiseuille Law.***

$\tau_{wall} = -\mu \left(\frac{dv}{dr}\right)|_{\{r=R\}} = \Delta P R/(2L) = 4\mu v_{mean}/R\ [Pa]$.

$dv/dr = -(\Delta P/2\mu L)r = -(2v_{max}/R^2)r$ $(velocity\ gradient)$.

The velocity gradient (and hence shear stress) is maximum at the wall (r = R) and zero at the centreline. This is why viscous degradation and erosion occur preferentially at the pipe wall.

The friction factor (Darcy-Weisbach): f = 64/Re where $Re = \rho v_{mean}(2R)/\mu$ is the Reynolds number. Flow is laminar for Re < 2300 and turbulent for Re > 4000. The pressure drop formula: $\Delta P = fL\rho v_{mean}^2/(2 \times 2R) = 32\mu L v_{mean}/R^2$.

8.3.4 Bernoulli's Equation and Flow Measurement

Bernoulli's equation for steady, inviscid, incompressible flow along a streamline:

$$\boldsymbol{p + \tfrac{1}{2}\rho v^2 + \rho g z = constant\ along\ a\ streamline}$$

Applied between a stagnation point (v = 0) and a free-stream point at velocity v:

$$\boldsymbol{p_{stag} = p_{static} + \tfrac{1}{2}\rho v^2 \rightarrow v = \sqrt{(2\Delta p/\rho)}}$$

The Pitot tube uses this principle to measure local velocity from the stagnation pressure difference Δp. The Venturi meter measures flow rate by applying Bernoulli between the upstream section (1) and throat (2):

$$\tfrac{1}{2}\rho v_1^2 + p_1 = \tfrac{1}{2}\rho v_2^2 + p_2$$

Combined with continuity $A_1 v_1 = A_2 v_2$:

$$\boldsymbol{Q = C_d\, A_2 \cdot \sqrt{(2\Delta p/\rho)} / \sqrt{(1 - (A_2/A_1)^2)}\ [m^3/s]}$$

Sensitivity of flow measurement: $dQ/dp = C_d A_2/(\sqrt{(2\rho\Delta p)} \cdot \sqrt{(1-\beta^4)})\ where\ \beta = D_2/D_1$. This shows that accuracy is poor at very low differential pressures (large dQ/dp) — a practical limitation of differential-pressure flow meters.

8.3.5 Optimal Pipe Diameter — Economic Calculus

For a pipeline delivering volumetric flow rate Q over length L, two competing costs exist: capital cost of pipe material (increases with D) and pumping energy cost (decreases

strongly with D, since ΔP ∝ 1/D⁵ from H-P law for laminar flow, or ∝ 1/D^5 approximately for turbulent flow).

For turbulent flow using the Darcy-Weisbach equation with f ≈ constant:

$$\Delta P = fL\rho v_{mean}^2/(2D) = fL\rho/(2D) \times (4Q/\pi D^2)^2 = 8fL\rho Q^2/(\pi^2 D^5)$$

Annual pumping cost: $C_{pump} = k_E \times Q \times \Delta P \times H_{annual}/\eta$ ($where\ k_E = energy\ cost\ [\$/J], H_{annual} = annual\ operating\ hours, \eta = pump\ efficiency$). This is proportional to $D^{\{-5\}}$.

Annual capital cost (annualised pipe cost): $C_{pipe} = k_C \times D$ (pipe cost per metre of diameter × length, annualised over design life).

$$\boldsymbol{C_{total}(D) = k_1 D + k^2 D^{\{-5\}}}$$

Minimising with respect to D:

$$dC_{total}/dD = k_1 - 5k^2 D^{\{-6\}} = 0$$

$$\boldsymbol{D^* = \left(\frac{5k^2}{k^1}\right)^{\left\{\frac{1}{6}\right\}}}$$

Verify minimum: $d^2C/dD^2 = 30k^2 D^{\{-7\}} > 0$ ✓. The optimal diameter scales with $Q\text{^}\left\{\frac{1}{3}\right\}$ for turbulent flow (since k₂ ∝ Q²), a useful rule of thumb for pipeline engineering.

✦ Worked Example 8.3.1 — Draining Water Tower

Cylindrical tower: $D = 8\ m, h_0 = 12\ m. Orifice: d_o = 150\ mm, C_d = 0.62.$

$$A_t = \pi(4)^2 = 50.27\ m^2;\ A_o = \pi(0.075)^2 = 0.01767\ m^2$$

(a) Initial Q_0: $v_{jet} = 0.62\sqrt{(2 \times 9.81 \times 12)} = 0.62 \times 15.34 = 9.511\ m/s$

$$Q_0 = 0.01767 \times 9.511 = 0.1680\ m^3/s = 168.0\ L/s$$

(b) $h(t) = [\sqrt{12} - (0.62 \times 0.01767/50.27)\sqrt{(9.81/2)} \times t]^2$

$$= [3.464 - (0.0001093 \times 2.215)t]^2$$

$$= [3.464 - 2.420 \times 10^{-4} \times t]^2\ [h\ in\ m, t\ in\ seconds]$$

(c) Time to drain from 12 m to 3 m:

$$\sqrt{h_0} - \sqrt{h} = 3.464 - 1.732 = 1.732$$

$$t = 1.732 / 2.420 \times 10^{-4} = 7157 / 2 \ldots let\ us\ use\ the\ formula:$$

$$t = (A_t / C_d\ A_o) \times (2/\sqrt{(2g)}) \times (\sqrt{h_0} - \sqrt{3})$$

$$= (50.27/0.01096) \times (2/4.429) \times (3.464 - 1.732)$$

$$= 4587 \times 0.4516 \times 1.732 = 3590\,s \approx 59.8\,min$$

(d) Total drain time: $t = (A_t/C_d\,A_o)\sqrt{(2h_0/g)} = 4587 \times \sqrt{(24/9.81)} = 4587 \times 1.564 = 7174\,s \approx 119.6\,min$

(e) Flow rate at h = 3 m:

$$Q = 0.62 \times 0.01767 \times \sqrt{(2 \times 9.81 \times 3)} = 0.01096 \times 8.668 = 0.08404\,m^3/s$$
$$= 84.0\,L/s$$

$Note: Q(h = 3)/Q(h = 12) = \sqrt{(3/12)} = 0.500$ — *halved as expected* ($Q \propto \sqrt{h}$).

✦ Worked Example 8.3.2 — Industrial Oil Pipeline: Laminar Flow Analysis

Oil: μ = 0.048 Pa·s, ρ = 870 kg/m³. Pipe: D = 25 mm, L = 4 m, ΔP = 12,000 Pa.

R = 0.0125 m

(a) $v_{max} = \Delta PR^2/(4\mu L) = 12000 \times (0.0125)^2/(4 \times 0.048 \times 4) = 12000 \times 1.5625 \times 10^{-4}/0.768$

$$= 1.875/0.768 = 2.441\,m/s$$

$$v(r) = 2.441[1 - (r/0.0125)^2]\,m/s$$

(b) $v_{mean} = v_{max}/2 = 1.220\,m/s$

$$Q = \pi R^2 v_{mean} = \pi(0.0125)^2 \times 1.220 = 5.987 \times 10^{-4}\,m^3/s = 0.599\,L/s$$

(c) $Re = \rho v_{mean}(2R)/\mu = 870 \times 1.220 \times 0.025/0.048 = 26.535/0.048 = 553 < 2300$ ✓ Laminar

(d) $\tau_{wall} = \Delta PR/(2L) = 12000 \times 0.0125/8 = 18.75\,Pa$

$Check: \tau_{wall} = 4\mu v_{mean}/R = 4 \times 0.048 \times 1.220/0.0125 = 0.2342/0.0125 = 18.73\,Pa$ ✓

(e) Radius where $v = v_{mean} = 1.220\,m/s$:

$$2.441(1 - r^2/R^2) = 1.220 \rightarrow r^2/R^2 = 0.5 \rightarrow r = R/\sqrt{2} = 0.00884\,m = 8.84\,mm$$

$$(Always\ at\ r = R/\sqrt{2}\ for\ Poiseuille\ flow —\ a\ universal\ result)$$

(f) Velocity gradient (shear rate) at r = 8 mm:

$$dv/dr = -2v_{max} \cdot r/R^2 = -2 \times 2.441 \times 0.008/(0.0125)^2$$
$$= -0.03906/1.5625 \times 10^{-4} = -250.0\,s^{-1}$$

Shear stress at $r = 8\,mm$: $\tau = \mu|dv/dr| = 0.048 \times 250 = 12.0\,Pa$

✦ Worked Example 8.3.3 — Venturi Flow Meter Calibration

Water (ρ = 1000 kg/m³) flows through a Venturi: D1 = 100 mm, D2 = 50 mm (throat).

Measured differential pressure: Δp = 18,000 Pa. $C_d = 0.98$.

Find: (a) throat velocity (b) flow rate (c) sensitivity dQ/dΔp.

(a) $\beta = D_2/D_1 = 0.5;\ \beta^4 = 0.0625$

$$A_1 = \pi(0.05)^2 = 8.854 \times 10^{-3}\,m^2;\ A_2 = \pi(0.025)^2 = 1.963 \times 10^{-3}\,m^2$$

$$v_2 = \sqrt{(2\Delta p/\rho)} \,/\, \sqrt{(1-\beta^4)} = \sqrt{(36)}/\sqrt{(0.9375)} = 6.000/0.9683 = 6.196\,m/s$$

(b) $Q = C_d\,A_2\,v_2 = 0.98 \times 1.963 \times 10^{-3} \times 6.196 = 0.98 \times 1.217 \times 10^{-2} = 0.01193\,m^3/s = 11.93\,L/s.$

(c) $dQ/d\Delta p$: $Q = C_d\,A_2\,\sqrt{(2\Delta p/\rho)}/\sqrt{(1-\beta^4)} = C_d\,A_2/\sqrt{(\rho(1-\beta^4))} \times \sqrt{2} \times \left(\frac{1}{2}\right)\Delta p^{\{-\frac{1}{2}\}}$

$$= Q/(2\Delta p) = 0.01193/(2 \times 18000) = 3.314 \times 10^{-7}\,m^3/s\ per\ Pa$$

$$= 0.3314\,L/s\ per\ kPa$$

At low Δp, this sensitivity is high — measurement uncertainty in pressure becomes critical.

(d) Re check at throat: $Re = \rho v_2 D_2/\mu = 1000 \times 6.196 \times 0.05/0.001 = 309{,}800 >>$ 4000 ✓ Turbulent

📝 Exercises 8.3 — Fluid Dynamics

8.3.1 A conical tank (apex down, half-angle 30°, h0 = 2 m) drains through $A_o = 5 \times 10^{-4}\,m^2$

($C_d = 0.61$). Derive h(t) for the cone. Find drain time. Compare to cylinder with same V.

8.3.2 Water (μ = 0.001 Pa·s, ρ = 1000 kg/m³) flows in D = 50 mm pipe, L = 10 m, Q = 2 L/s.

(a) Check if laminar (b) if turbulent, estimate ΔP using $f = 0.316Re^{\{-0.25\}}$ (Blasius).

8.3.3 Show: for Poiseuille flow, the radius at which $v = v_{mean}\ is\ r = R/\sqrt{2}$.

Δp = 25 kPa. Find Q and v_{throat}.

8.3.5 Show that doubling pipe radius increases Q by factor 16 for fixed ΔP, μ, L.

How much must radius increase to double Q (all else fixed)?

8.3.6 A sump pump discharges Q = 0.03 m³/s through L = 50 m of horizontal pipe.

Pipe material: PVC (f = 0.018). Allowable head loss = 5 m. Find D.

8.4 Electrical Circuit Analysis (Rate-Based)

8.4.1 Constitutive Relations and KVL/KCL

Energy storage elements in circuits — capacitors and inductors — are characterised by constitutive relations that inherently involve the derivative:

Element	***Symbol***	***Constitutive Relation***	***Energy Stored***	***Units***
$Resistor$	R	$v = Ri\ (Ohm's\ Law)$	$0\ (dissipates)$	Ω
$Capacitor$	C	$i = C\,dv/dt$	$E_C = ½Cv^2$	F
$Inductor$	L	$v = L\,di/dt$	$E_L = ½Li^2$	H

Kirchhoff's Voltage Law (KVL): The algebraic sum of voltages around any closed loop is zero. Kirchhoff's Current Law (KCL): The algebraic sum of currents entering any node is zero. Combining KVL/KCL with the constitutive relations produces ODEs governing the circuit's time response. The order of the ODE equals the number of independent energy storage elements.

The concept of time constant τ appears in first-order circuits. For an RC circuit: τ = RC [s]. For an RL circuit: $\tau = L/R\ [s]$. At t = τ, the response has decayed to $e^{\{-1\}} \approx 36.8\%$ of its initial value, or risen to $(1 - e^{\{-1\}}) \approx 63.2\%$ of its final value.

8.4.2 RC Circuits — Charging and Discharging

Charging (source V_s, $initial\ v_C(0) = 0$)

KVL around the series RC loop: $V_s = v_R + v_C = Ri + v_C. Since\ i = C(dv_C/dt)$:

$$\boldsymbol{RC \cdot dv_C/dt + v_C = V_s \Longrightarrow \tau = RC}$$

Rearranging: $dv_C/dt = (V_s - v_C)/\tau$. This is separable. Integrating with $v_C(0) = 0$:

$$\boldsymbol{v_C(t) = V_s(1 - e^{\{-\frac{t}{\tau}\}})}$$

$$i(t) = C\,dv_C/dt = \left(\frac{V_s}{R}\right)e^{\left\{-\frac{t}{\tau}\right\}}$$

The slew rate (rate of voltage rise) is maximum at $t = 0$: $dv_C/dt|_0 = V_s/\tau = V_s/(RC)$.

Discharging (no source, initial $v_C(0) = V_0$)

$$v_C(t) = V_0\ e^{\left\{-\frac{t}{\tau}\right\}}\ i(t) = -(V_0/R)\ e^{\left\{-\frac{t}{\tau}\right\}}$$

Energy dissipated in R during full discharge (t: 0 → ∞):

$$E_R = \int_0^\infty i^2 R\,dt = \int_0^\infty (V_0{}^2/R)\quad e^{\left\{-\frac{2t}{\tau}\right\}}\,dt = (V_0{}^2/R)(\tau/2) = \tfrac{1}{2}CV_0{}^2\ [J]$$

All stored energy ½CV0² is dissipated in the resistor — regardless of R! This counter-intuitive result follows directly from the integral, and explains why the energy efficiency of charging a capacitor through a resistor from a voltage source is always 50%.

8.4.3 RL Circuits — Energisation and De-energisation

Series RL with supply V_s. $By\ KVL$: $V_s = Ri + L(di/dt)$. Rearranging:

$$(L/R)\,di/dt + i = V_s/R \implies \tau = L/R$$

$$i(t) = \left(\frac{V_s}{R}\right)\left(1 - e^{\left\{-\frac{t}{\tau}\right\}}\right) v_L(t) = L\frac{di}{dt} = V_s\,e^{\left\{-\frac{t}{\tau}\right\}}$$

At t = 0: all supply voltage appears across L ($v_L = V_s, i = 0$). This is because an inductor acts as an open circuit instantaneously — it opposes any instantaneous change in current. At $t \to \infty$: $v_L \to 0, i \to V_s/R$ (inductor is a short circuit at DC).

When the circuit is broken suddenly (switch opens with current I0 flowing), the inductor attempts to maintain I0. If the switch creates an open circuit, $v_L \to \infty$ in theory. In practice, arcing occurs at the switch contacts. The energy ½LI0² must be dissipated or diverted. Flyback diodes in relay circuits serve this protective purpose.

De-energisation through R (switch reconfigures to discharge path): $i(t) = I_0\ e^{\left\{-\frac{t}{\tau}\right\}}$, $v_L = -LI_0/\tau \cdot e^{\left\{-\frac{t}{\tau}\right\}} = -RI_0\ \ e^{\left\{-\frac{t}{\tau}\right\}}$ (negative, opposing current flow).

8.4.4 RLC Series Circuit — Second-Order Response

For a series RLC circuit (KVL with capacitor voltage as state variable):

$$L\,d^2i/dt^2 + R\,di/dt + i/C = dv_s/dt$$

For the natural response ($v_s = 0$ after initial conditions imposed):

$$d^2i/dt^2 + 2\alpha\,di/dt + \omega_0{}^2\,i = 0$$

where α = R/(2L) [Np/s] is the neper (damping) frequency and ω₀ = 1/√(LC) [rad/s] is the resonant frequency. The damping ratio $\zeta = \alpha/\omega_0 = (R/2)\sqrt{(C/L)}$.

Characteristic equation: $s^2 + 2\alpha s + {\omega_0}^2 = 0, giving\ s_{\{1,2\}} = -\alpha \pm \sqrt{(\alpha^2 - {\omega_0}^2)}$.

Case	*Condition*	*Roots*	*Response Form*
Underdamped	$\zeta < 1, \alpha < \omega_0$	$s = -\alpha \pm j\omega_d$	$e^{\{-\alpha t\}}(A\ cos\omega_d\ t + B\ sin\omega_d\ t)$
Critically damped	$\zeta = 1, \alpha = \omega_0$	$s = -\alpha\ (double)$	$(A + Bt)e^{\{-\alpha t\}}$
Overdamped	$\zeta > 1, \alpha > \omega_0$	$s_{1,2}\ real, negative$	$Ae^{\{s_1\ t\}} + Be^{\{s_2 t\}}$

where $\omega_d = \sqrt{({\omega_0}^2 - \alpha^2)} = \omega_0\sqrt{(1 - \zeta^2)}$ is the damped natural frequency. The electrical-mechanical analogy is exact: mass m ↔ inductance L, damping c ↔ resistance R, stiffness k ↔ 1/C, displacement x ↔ charge q, velocity v ↔ current i, force F ↔ voltage v_s.

8.4.5 Differentiator and Integrator Op-Amp Circuits

An ideal operational amplifier with differential input and very high open-loop gain produces two important derivative-based circuits when combined with R and C:

Ideal Differentiator

Input capacitor C, feedback resistor R. The ideal output (assuming infinite op-amp gain and bandwidth):

$$\boldsymbol{v_{out}(t) = -RC \cdot dv_{in}/dt}$$

The output is proportional to the derivative of the input. Frequency response: $gain = \omega RC$ (increases without bound with frequency). This makes the ideal differentiator noise-amplifying at high frequencies, since electronic noise is wideband. A practical differentiator adds a series input resistor R₁ to roll off gain: $gain = -j\omega RC/(1 + j\omega R_1 C)$.

Ideal Integrator

Input resistor R, feedback capacitor C. The ideal output:

$$\boldsymbol{v_{out}(t) = -(1/RC)\int_0^t v_{in}\, dt' \quad + v_{out}(0)}$$

Frequency response: $gain = 1/(\omega RC)$ (decreases with frequency — low-pass behaviour). DC offset in $v_{in}\ causes\ v_{out}$ to grow linearly (integrate a constant → ramp). Practical integrators add a large feedback resistor R_f parallel to C to prevent DC saturation.

✦ Worked Example 8.4.1 — Camera Flash: RC Charging Circuit

$R = 2500\,\Omega, C = 2000\,\mu F = 0.002\,F, V_s = 300\,V. \tau = RC = 5.0\,s.$

(a) $v_C(t) = 300\left(1 - e^{\left\{-\frac{t}{5}\right\}}\right) V;\ i(t) = 0.12e^{\left\{-\frac{t}{5}\right\}} A$

(b) $v_C(2) = 300(1 - e^{\{-0.4\}}) = 300(1 - 0.6703) = 98.91\,V$

$$i(2) = 0.12e^{\{-0.4\}} = 0.12 \times 0.6703 = 80.44\,mA$$

(c) Time to reach 270 V: $e^{\left\{-\frac{t}{5}\right\}} = 30/300 = 0.10$

$$t = -5\,ln(0.10) = 5 \times 2.3026 = 11.51\,s$$

(d) Peak slew rate ($at\ t = 0$): $dv_C/dt|_0 = V_s/\tau = 300/5 = 60\,V/s$

(e) Energy stored at full charge: $E = ½CV_s^2 = ½ \times 0.002 \times 90000 = 90\,J$

Flash duration ~1 ms: Peak power = 90/0.001 = 90,000 W = 90 kW

(f) Energy dissipated in R during full charging: E_R = ½CV_s² = 90 J

(Equal to energy stored — charging efficiency is always 50% for RC circuit)

(g) At t = 3τ = 15 s: $v_C = 300(1 - e^{\{-3\}})$ = 300×0.9502 = 285.1 V (95% charged)

At t = 5τ = 25 s: $v_C = 300(1 - e^{\{-5\}})$ = 300×0.9933 = 297.9 V (99.3% charged)

✦ Worked Example 8.4.2 — Industrial Electromagnet: RL Circuit

R = $4\,\Omega, L = 0.5\,H, V_s = 24\,V. \tau = L/R = 0.125\,s. I_{ss} = V_s/R = 6\,A.$

(a) $i(t) = 6(1 - e^{\left\{-\frac{t}{0.125}\right\}}) = 6(1 - e^{\{-8t\}})\,A$

$$v_L(t) = 24e^{\{-8t\}}\,V$$

(b) $i(0.2) = 6(1 - e^{\{-1.6\}}) = 6(1 - 0.2019) = 4.789\,A$

$$v_L(0.2) = 24e^{\{-1.6\}} = 24 \times 0.2019 = 4.845\,V$$

$v_R(0.2) = Ri(0.2) = 4 \times 4.789 = 19.16\,V;\ check$: $19.16 + 4.84 = 24\,V$ ✓

(c) Time for i = 5 A:

$5 = 6(1 - e^{\{-8t\}}) \rightarrow e\text{^}\{-8t\} = 1/6 \rightarrow t = ln(6)/8 = 1.7918/8 = 0.2240\,s$

(d) Power dissipated at $t = 0.1\,s$: $i(0.1) = 6(1 - e^{\{-0.8\}}) = 6 \times 0.5507 = 3.304\,A$

$$P_R = i^2R = (3.304)^2 \times 4 = 43.67\,W$$

$$P_L = v_L \times i = (24e^{\{-0.8\}}) \times 3.304 = (24 \times 0.4493) \times 3.304 = 10.78 \times 3.304 = 35.62\,W$$

$P_{total} = V_s \times i = 24 \times 3.304 = 79.30\,W$ ($check$: $43.67 + 35.62 = 79.29\,W$ ✓)

(e) Energy stored in inductor at steady state: $E_L = ½LI_{ss}^2 = ½ \times 0.5 \times 36 = 9\,J$

This energy is released (and can cause voltage spikes) when the circuit is broken.

✦ Worked Example 8.4.3 — RLC Circuit: Underdamped Natural Response

$R = 40\,\Omega, L = 0.1\,H, C = 100\,\mu F. Initial: v_C(0) = 100\,V, i(0) = 0.$

(a) $\omega_0 = 1/\sqrt{(LC)} = 1/\sqrt{(0.1\times10^{-4})} = 1/\sqrt{(10^{-5})} = 316.2$ rad/s

$$\alpha = R/(2L) = 40/0.2 = 200\,Np/s$$

$$\zeta = \alpha/\omega_0 = 200/316.2 = 0.6325 < 1 \rightarrow UNDERDAMPED$$

(b) $\omega_d = \omega_0\sqrt{(1-\zeta^2)} = 316.2\sqrt{(1-0.40)} = 316.2\times0.7746 = 244.9$ rad/s

$$f_d = \omega_d/(2\pi) = 39.0\,Hz;\ T_d = 1/39.0 = 25.64\,ms$$

(c) $v_C(t) = e^{\{-200t\}}$[A cos(244.9t) + B sin(244.9t)]

$$v_C(0) = A = 100\,V$$

$$i(0) = C(dv_C/dt)|_0 = 0: dv_C/dt|_0 = -\alpha A + B\omega_d = 0$$

$$B = \alpha A/\omega_d = 200 \times 100/244.9 = 81.67\,V$$

$$v_C(t) = e^{\{-200t\}}[100\,cos(244.9t) + 81.67\,sin(244.9t)]\,V$$

$$Amplitude: C_{amp} = \sqrt{(100^2 + 81.67^2)} = \sqrt{(10000 + 6670)} = \sqrt{16670} = 129.1\,V$$

$$Phase: \varphi = arctan(81.67/100) = 39.26°$$

$$v_C(t) = 129.1\,e^{\{-200t\}}\,cos(244.9t - 39.26°)\,V$$

(d) Time for first zero-crossing of v_C: $cos(244.9t - 39.26°) = 0$

$$244.9t = 39.26° + 90° = 129.26° = 2.257\,rad \rightarrow t = 9.22\,ms$$

(e) Energy at $t = 0$: $E_0 = ½Cv_C(0)^2 = ½ \times 10^{-4} \times 10000 = 0.5\,J$

$$Energy\ at\ t = T_d/2 = 12.82\,ms:$$

$$v_C(12.82ms) \approx 129.1\,e^{\{-200\times0.01282\}}\,cos(\pi - 39.26°)$$

$$= 129.1 \times e^{\{-2.564\}} \times (-cos39.26°)$$

$$= 129.1 \times 0.07688 \times (-0.7746) = -8.693\,V$$

$$E(T_d/2) \approx ½ \times 10^{-4} \times 59.2$$

$$= 2.96 \times 10^{-3}\,J\ (99.4\%\ of\ energy\ dissipated\ in\ half\ cycle!)$$

✦ Worked Example 8.4.4 — Op-Amp Differentiator: Rate-of-Change Sensor

Differentiator: R = 10 kΩ, C = 0.1 μF. RC = 1×10^{-3} s.

Input: $v_{in}(t) = 3\,sin(2000\pi t)\,V\ (f = 1000\,Hz, \omega = 2000\pi\,rad/s)$.

(a) $v_{out} = -RC\,dv_{in}/dt = -10^{-3} \times 3 \times 2000\pi\,cos(2000\pi t)$

$$= -6\pi\,cos(2000\pi t) = -18.85\,cos(2000\pi t)\,V$$

(b) Amplitude: $|V_{out}| = \omega RC \times |V_{in}| = 2000\pi \times 10^{-3} \times 3 = 6\pi \approx 18.85\,V$

(6.28× amplification at 1 kHz)

(c) Phase: output leads input by 90° (cos = sin shifted +90°, then –ve adds 180°)

Net phase shift = +90° lead relative to input signal

(d) At f = 100 Hz: $|V_{out}|$ = 200π×10^{-3}×3 = 0.6π ≈ 1.885 V (lower amplification)

At f = 5000 Hz: $|V_{out}|$ = 10000π×10^{-3}×3 = 30π ≈ 94.25 V — CLIPPING would occur

Practical differentiators add $R_{in} \approx 1\,k\Omega$ in series to limit gain at high f.

(e) If input is a ramp: $v_{in} = 2t\,V, then\ v_{out} = -RC \times 2 = -2 \times 10^{-3} = -0.002\,V\ (DC)$

Differentiator converts ramp to constant — used in velocity sensing of position encoders.

📝 Exercises 8.4 — Electrical Circuit Analysis

8.4.1 RC: R = 47 kΩ, C = 10 μF, V_s = 9 V. Find: (a) τ (b) $v_C(t)$ (c) t when $v_C = 6.3\,V$

(d) energy stored at t = τ (e) charging efficiency.

8.4.2 RL: R = 8 Ω, L = 200 mH, $V_s = 12\,V$. Find: (a) i(t) (b) $v_L(t)$ (c) P_R at t = 50 ms

(d) time for i to reach 90% of I_{ss} (e) energy stored at steady state.

8.4.3 Series RLC: R = 100 Ω, L = 50 mH, C = 20 μF.

Find ω0, α, ζ. Classify the response. Write i(t) general form for natural response.

8.4.4 Derive the differential equation for v_C in a parallel RLC driven by current source $i_s(t)$.

Show it has the same characteristic equation as the series case (with appropriate α and ω0).

8.4.5 An op-amp integrator: R = 100 kΩ, C = 1 μF, v_in = 5 V (step), $v_{out}(0) = 0$.

Find $v_{out}(t)$ and the time for $|v_{out}|$ to reach the supply limit of ±12 V.

8.4.6 Show that the energy dissipated in R when charging a capacitor C from 0 to V_s through R is always $\frac{1}{2}CV_s^2$, independent of R. Interpret physically.

8.5 Structural Analysis Problems

8.5.1 Stress, Strain, and the Tangent Modulus

In a uniaxial tensile test, engineering stress σ and engineering strain ε are:

$$\boldsymbol{\sigma = F/A_0\ [Pa]\ \varepsilon = (L - L_0)/L_0\ \textbf{[dimensionless]}}$$

In the elastic region, Hooke's Law gives $\sigma = E\varepsilon$, with Young's modulus E [GPa]. Beyond the yield point, the tangent modulus E_t — the local slope of the σ-ε curve — is the key material parameter:

$$\boldsymbol{E_t(\varepsilon) = d\sigma/d\varepsilon}$$

In the elastic range, $E_t = E\ (constant)$. For a Ramberg-Osgood power-law material (widely used for aluminium and titanium alloys):

$$\boldsymbol{\varepsilon = \sigma/E + \left(\frac{\sigma}{K}\right)^n}$$

Differentiating implicitly with respect to σ:

$$d\varepsilon/d\sigma = 1/E + \left(\frac{n}{K}\right)\left(\frac{\sigma}{K}\right)^{\{n-1\}}$$

$$\boldsymbol{E_t = \left[\frac{1}{E} + \left(\frac{n}{K}\right)\left(\frac{\sigma}{K}\right)^{\{n-1\}}\right]^{\{-1\}}}$$

Note: $E_t \leq E$ always, with equality at σ = 0. As σ increases, E_t decreases — softening. This is why structures loaded beyond yield are analysed using nonlinear finite element methods.

True stress σ_{true} and true strain ε_{true} (logarithmic strain) are related to engineering values:

$$\boldsymbol{\sigma_{true} = \sigma(1 + \varepsilon)[before\ necking]\varepsilon_{true} = ln(1 + \varepsilon)}$$

$$\boldsymbol{\frac{d\sigma_{true}}{d\varepsilon_{true}} = \sigma_{true} at\ necking\ (Considère\ criterion)}$$

8.5.2 Flexure Formula and Shear Stress Distribution

For a beam in pure bending, the flexure (bending stress) formula is derived from the assumption that plane sections remain plane (Euler-Bernoulli hypothesis) and Hooke's Law:

$$\boldsymbol{\sigma(y) = My/I\ [Pa]}$$

where M is the applied bending moment, y is the distance from the neutral axis, and $I = \int y^2 dA$ is the second moment of area. The linear variation means:

$$\boldsymbol{d\sigma/dy = M/I = constant}$$

Maximum bending stress at extreme fibre y = c:

$$\boldsymbol{\sigma_{max} = \frac{Mc}{I} = \frac{M}{Z}\ where\ Z = I/c\ [m^3]\ is\ the\ elastic\ section\ modulus}$$

Shear stress distribution in a rectangular section (width b, depth d):

Applying horizontal equilibrium to a layer cut at height y from the neutral axis (Jouravski's method):

$$\tau(y) = \frac{VQ}{Ib}\ where\ Q = \int_y^{\frac{d}{2}} y'\, b\ \ dy' = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\boldsymbol{\tau(y) = \frac{V\left(\frac{d^2}{4} - y^2\right)}{2I}\ [parabolic\ distribution]}$$

Maximum shear stress at neutral axis (y = 0): $\tau_{max} = Vd^2/(8I) = 3V/(2A)$ (for rectangle). The shear stress is zero at the extreme fibres (y = ±d/2) — confirmed by $d\tau/dy = 0$ at $y = 0$ (maximum) and $\tau = 0\ at\ y = \pm d/2$.

For an I-section, the shear stress is constant across the flange (small contribution) and varies parabolically in the web. The abrupt change at the flange-web junction is a stress concentration that can cause horizontal shear failure in timber beams.

8.5.3 Euler Column Buckling — Derivation from the Beam-Column Equation

A slender elastic column under axial compressive load P develops lateral deflection y(x) governed by the beam-column differential equation. Taking x along the column axis and y as lateral displacement:

$$\boldsymbol{EI\ d^2y/dx^2 + P\ y = 0}$$

This is derived from the bending moment $M = EI\,y''$ and the moment equilibrium $M = -Py$ at any cross-section (for small deflections). Substituting $\lambda^2 = P/(EI)$:

$$\boldsymbol{y'' + \lambda^2 y = 0}$$

General solution: $y(x) = A\,sin(\lambda x) + B\,cos(\lambda x)$. For pin-pin boundary conditions:

$$y(0) = 0 \rightarrow B = 0$$

$$y(L) = 0 \rightarrow A\,sin(\lambda L) = 0$$

Non-trivial solution requires $sin(\lambda L) = 0, so\ \lambda L = n\pi\ (n = 1, 2, 3, \dots)$. The critical (lowest) load corresponds to n = 1:

$\boldsymbol{P_{cr} = \pi^2 EI/L^2 \leftarrow}$ **Euler's Critical Load (pin-pin)**

The first buckled mode shape is $y(x) = A\,sin(\pi x/L)$ — a half sine wave. The second derivative of P_{cr} with respect to I: $\frac{dP_{cr}}{dI} = \frac{\pi^2 E}{L^2} > 0$ — increasing the second moment of area always increases buckling resistance. This drives the design of I-sections and hollow tubes (material placed as far from the neutral axis as possible).

Slenderness ratio $\lambda_s = \frac{L_e}{r}\ where\ r = \sqrt{(I/A)}$ is the radius of gyration and L_e is the effective length. Euler's formula is valid only when the critical stress $\sigma_{cr} = P_{cr}/A$ is below the proportional limit of the material:

$\boldsymbol{\sigma_cr = \pi^2 E/\lambda_s^2 < \sigma_P L}$

End Conditions	***Effective Length L_e***	***P_cr***
$Pin - Pin$	L	$\pi^2 EI/L^2$
$Fixed - Free\ (cantilever)$	$2L$	$\pi^2 EI/(4L^2)$
$Fixed - Pin$	$\approx 0.699L$	$\approx 2.046\,\pi^2 EI/L^2$
$Fixed - Fixed$	$0.5L$	$4\pi^2 EI/L^2$

8.5.4 Combined Loading — Superposition

When a structural member is simultaneously subject to axial force P, bending moment M, and transverse shear V, the stresses are superimposed (valid in the elastic range):

$$\boldsymbol{\sigma_{total}(y) = \sigma_{axial} + \sigma_{bending} = P/A \pm My/I}$$

The critical cross-section and critical fibre must be identified by examining all load combinations. The von Mises yield criterion (for general combined stress states):

$$\boldsymbol{\sigma_V M = \sqrt{\sigma_{total}^2 + 3\tau^2} \leq \frac{f_y}{n_s}}$$

where n_s is the safety factor. Differentiating with respect to y to locate the maximum combined stress:

$$\frac{d(\sigma_V M)}{dy} = \frac{\sigma_{total} \times \frac{d\sigma_{total}}{dy} + 3\tau \times \frac{d\tau}{dy}}{\sigma_V M} = 0$$

Setting numerator to zero: $\sigma_{total} \times (M/I) = 3\tau \times (Vy)/(I)$ — this yields the y-location of maximum von Mises stress, which generally does not coincide with either the extreme fibre (σ_{max}) or the neutral axis (τ_{max}).

8.5.5 Deflection Optimisation — Stiffness-Weight Trade-off

For a simply supported beam of rectangular cross-section (b × d), span L, under central load P:

$$\boldsymbol{\delta_{max} = \frac{PL^3}{48EI} = \frac{PL^3}{48E \times \frac{bd^3}{12}} = \frac{PL^3}{4Ebd^3}}$$

Subject to the constraint that the cross-sectional area $A = bd = constant$ (fixed weight per unit length), we may choose the aspect ratio d/b to minimise δ:

With $b = A/d: \delta = PL^3/(4E(A/d)d^3) = PL^3/(4EAd^2). Since\ A\ is\ fixed, \delta \propto 1/d^2$ — minimise δ by maximising d! A deep, narrow beam is stiffer in bending than a shallow, wide beam of the same cross-sectional area.

However, a very narrow beam may fail by lateral-torsional buckling. The critical moment for lateral-torsional buckling scales as $M_{cr} \propto \frac{\sqrt{EI_y\, GJ}}{L} where\ I_y$ is the weak-axis second moment and J is the Saint-Venant torsion constant. Optimising between bending stiffness (deep beam) and lateral stability (wide beam) is a practical engineering trade-off governed by derivative analysis.

✦ Worked Example 8.5.1 — Wide-Flange Beam: Complete Stress Analysis

$W200 \times 59\ steel: I = 200 \times 10^6\ mm^4, d = 210\ mm, b_f = 205\ mm, t_f = 14.2\ mm, t_w = 9.1\ mm.$

$A = 7550\ mm^2. Loading\ at\ section: M = 180\ kN \cdot m, V = 250\ kN.$

(a) $\sigma_{max} = Mc/I = (180 \times 10^6)(105)/(200 \times 10^6) = 94.5\ MPa$

$(f_y = 355\ MPa\ for\ S355\ steel \rightarrow utilisation = 94.5/355 = 26.6\%\ in\ bending)$

(b) $\frac{d\sigma}{dy} = \frac{M}{I} = 180 \times 10^6/200 \times 10^6 = 0.90\ MPa/mm$

(bending stress changes by 0.90 MPa for every mm from neutral axis)

(c) First moment Q at neutral axis:

$$Q_{flange} = b_f \times t_f \times (\frac{d}{2} - \frac{t_f}{2}) = 205 \times 14.2 \times 97.9 = 284{,}893\ mm^3$$

$$Q_{web_{half}} = t_w \times (d/2 - t_f)^2/2 = 9.1 \times (90.8)^2/2 = 37{,}513\ mm^3$$

$$Q_N\ A = 284{,}893 + 37{,}513 = 322{,}406\ mm^3$$

$$\tau_N A = VQ/(Ib) = (250{,}000 \times 322{,}406)/(200 \times 10^6 \times 9.1) = 44.29\ MPa$$

(d) τ at flange-web junction ($y = 90.8\ mm, web\ side, b = t_w = 9.1\ mm$):

$$\tau_{junc} = VQ_f/(It_w) = (250{,}000 \times 284{,}893)/(200 \times 10^6 \times 9.1) = 39.15\ MPa$$

$$\tau\ at\ flange\ side\ (b = b_f = 205\ mm): \tau = 39.15 \times 9.1/205 = 1.74\ MPa$$

(e) $\sigma_V M$ at flange-web junction ($\sigma = My_{junc}/I = 180 \times 10^6 \times 90.8/200 \times 10^6 = 81.7\ MPa$):

$$\sigma_V M = \sqrt{(81.7^2 + 3 \times 39.15^2)} = \sqrt{(6675 + 4600)} = \sqrt{11275} = 106.2\ MPa$$
$$< 355\ MPa\ \checkmark$$

✦ Worked Example 8.5.2 — SHS Column Buckling ($Pin - Pin$)

Square hollow section $150 \times 150 \times 8\ mm, S355\ steel\ (E = 200\ GPa, f_y = 355\ MPa), L = 4.5\ m.$

(a) $I = (b^4 - b_i^4)/12 = (150^4 - 134^4)/12 = (506.25 - 321.42) \times 10^6/12 = 15.40 \times 10^6\ mm^4$

$$A = 150^2 - 134^2 = 22500 - 17956 = 4544\ mm^2$$

$$r = \sqrt{(I/A)} = \sqrt{(15.40 \times 10^6/4544)} = \sqrt{3389} = 58.23\ mm$$

(b) $P_{cr} = \pi^2 EI/L^2 = \pi^2 \times 200{,}000 \times 15.40 \times 10^6/(4500^2)$

$$= 9.870 \times 3.080 \times 10^{12}/20.25 \times 10^6 = 1{,}501\ kN$$

(c) $\sigma_{cr} = P_{cr}/A = 1{,}501{,}000/4544 = 330.4\ MPa < f_y = 355\ MPa$

Elastic buckling governs ($\sigma_{cr} < f_y$) ✓

(d) Slenderness ratio $\lambda_s = \frac{L}{r} = 4500/58.23 = 77.3$

Limiting slenderness: $\lambda_{lim} = \pi\sqrt{(E/f_y)} = \pi\sqrt{(200{,}000/355)} = \pi \times 23.73 = 74.6$

Since 77.3 > 74.6, Euler formula is valid (just). ✓

(e) $dP_{cr}/dL = -2\pi^2 EI/L^3 = -2 \times 1{,}501/4500 = -0.667\ kN/mm$

Sensitivity: every 1 mm increase in column height reduces $P_{cr}\ by\ 0.667\ kN$.

A 100 mm height increase → $P_{cr}\ drops\ by\ 66.7\ kN\ (4.4\%\ reduction)$.

✦ Worked Example 8.5.3 — Crane Column: Combined Axial and Bending

I-section column: $A = 8400\ mm^2, I = 320 \times 10^6\ mm^4, c = 125\ mm.$

Loading: $P = 600\ kN\ (axial), M = 90\ kN \cdot m$ (lateral at this cross-section).

(a) $\sigma_{axial} = P/A = 600{,}000/8400 = 71.43\ MPa$ (uniform, compressive)

(b) $\sigma_{bending} = Mc/I = (90 \times 10^6)(125)/(320 \times 10^6) = 35.16\ MPa$

(±35.16 MPa on extreme fibres)

(c) Combined stresses (taking compression positive):

High-compression fibre: σ = 71.43 + 35.16 = 106.6 MPa (C) ← governs

Low-compression fibre: σ = 71.43 − 35.16 = 36.27 MPa (C)

Both fibres remain in compression — no tensile region exists.

(d) Neutral axis for combined loading (σ = 0, measuring y from NA):

$$-\frac{P}{A} + \frac{My}{I} = 0 \rightarrow y = (\frac{P}{A})(\frac{I}{M}) = 71.43 \times (320/90) = 71.43 \times 3.556 = 254\ mm$$

$$Since\ 254\ mm > c = 125\ mm, neutral\ axis\ is\ outside\ cross-section. \checkmark$$

(e) Utilisation ratio $UR = \sigma_{max}/f_y = 106.6/355 = 0.300$ (30% utilised in bending+axial)

Axial utilisation: $P/P_{cr}\ where\ P_{cr} = 1501\ kN\ (from\ 4.5\ m\ height)$: $600/1501 = 0.400$

Combined check (interaction formula): $P/P_{cr} + M/M_{cr} \leq 1.0$

$$M_{cr} = f_y \times Z = 355 \times (320 \times 10^6/125)/10^6 = 355 \times 2560 = 908.8\ kN \cdot m$$

0.400 + 90/908.8 = 0.400 + 0.099 = 0.499 < 1.0 ✓ (Column is adequate.)

✦ Worked Example 8.5.4 — Optimal Beam Section: Stiffness Maximisation

A simply supported timber beam, $span\ L = 5\ m$, carries $P = 10\ kN$ centrally.

$E = 12\,GPa$. Cross-section is rectangular $b \times d$ with area $A = bd = 20{,}000\,mm^2$ fixed.

Deflection limit $\delta_{max} = L/300 = 16.67\,mm$. Find the optimum aspect ratio d/b.

$$\delta = PL^3/(48EI) = PL^3/(4Ebd^3)\ (using\ I = bd^3/12)$$

With $b = A/d = 20{,}000/d$:

$$\delta = PL^3/(4E \times (20000/d) \times d^3) = PL^3/(4E \times 20000 \times d^2)$$
$$= PL^3/(80000Ed^2)$$

$$Setting\ \delta = \delta_allow = L/300:$$

$$L/300 = PL^3/(80000Ed^2)$$

$$d^2 = 300PL^2/(80000E) = 300 \times 10{,}000 \times 25 \times 10^6/(80{,}000 \times 12{,}000)$$
$$= 75 \times 10^9/960 \times 10^6 = 78.125 \times 10^3$$

$$d = 279.5\,mm;\ b = 20{,}000/279.5 = 71.6\,mm$$

$$Aspect\ ratio\ d/b = 279.5/71.6 = 3.903 \approx 4$$

$$Check: I = 71.6 \times (279.5)^3/12 = 71.6 \times 21.84 \times 10^6/12 = 130.4 \times 10^6\,mm^4$$

$$\delta = 10{,}000 \times (5000)^3/(48 \times 12{,}000 \times 130.4 \times 10^6)$$
$$= 10^4 \times 1.25 \times 10^{11}/(48 \times 1.565 \times 10^{12}) = 16.62\,mm$$
$$\approx 16.67\,mm\ \checkmark$$

Engineering note: Standard timber sections use $d/b \approx 3\ to\ 5$ — this analysis confirms

the practical wisdom. Wider beams ($d/b < 3$) are less stiff; deeper beams need LTB checks.

Exercises 8.5 — Structural Analysis

8.5.1 $W250 \times 89$ ($I = 395 \times 10^6\,mm^4, d = 260\,mm, A = 11{,}400\,mm^2$).

$$M = 240\,kN \cdot m, V = 180\,kN. Find: (a)\sigma_{max}\ (b)\ \frac{d\sigma}{dy}\quad (c)\tau_N\ A$$

(d) $\sigma_V M$ at the flange-web junction.

8.5.2 Hollow aluminium tube (OD = 100 mm, wall t = 8 mm, E = 70 GPa), L = 3 m, pin-pin.

Find: $(a)I \qquad (b)P_{cr} \qquad (c)\sigma_{cr} \qquad (d)\lambda_s\ \frac{(e)dP_{cr}}{dI}$.

8.5.3 Derive $\tau(y) = V(d^2/4 - y^2)/(2I)$ for a rectangular section $b \times d$.

Show $\tau_{max} = 3V/(2bd)$ *and confirm* $d(\tau)/dy = 0$ *at* $y = 0$.

8.5.4 A circular steel rod $(d = 25\,mm, E = 200\,GPa, f_y = 250\,MPa)$ *carries* $P = 50\,kN$ axial and M = 0.8 kN·m

Find: (a) σ_{max} (b) check yielding (c) find Euler load for L = 2 m.

8.5.5 Prove using calculus that for a fixed cross-sectional area A, the rectangular section that maximises I is the one with $d/b \to \infty$ $(infinitely\ deep)$. Why is this impractical? What constraint limits d/b in practice?

8.5.6 A 6 m timber beam (b = 100 mm, d = 300 mm, $E = 12\,GPa, L = 6\,m,$ simply supported) carries 12 kN centrally. Find: (a) I (b) σ_{max} (c) δ_{max} (d) check L/300 limit

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8.6 Open-Ended Engineering Design Problems

8.6.1 Purpose and Methodology

Open-ended design problems are the hallmark of authentic engineering practice. Unlike structured textbook exercises where a unique correct answer exists and the solution path is prescribed, open-ended problems require the engineer to: (1) translate a real-world performance requirement into a mathematical model; (2) identify and justify simplifying assumptions; (3) solve the resulting equations; (4) interpret results physically; (5) evaluate the design against criteria; and (6) propose improvements and identify limitations.

The problems below use differential calculus as the primary tool — through optimisation (dC/dx = 0), rate analysis (dT/dt, dv/ds), and sensitivity (dOutput/dParameter). Each solution framework demonstrates the complete engineering problem-solving process. A worked solution is provided for each problem, followed by extension questions.

✦ Design Problem 8.6.1 — Optimum Insulation: Cold Storage Facility

Facility: 1200 m^2 wall/roof area. Interior $T_i = -18°C$, Exterior $T_o = 38°C$ $(\Delta T = 56°C)$.

Layers: $h_i = 8\ W/m^2K$ | plywood 100mm k=0.15 | foam thickness t [m] k=0.025 | $h_e =$ 20 W/m²K. COP = 2.8. Electricity: \$0.14/kWh. Insulation cost: \$85/m³. Life 25 yr, i = 6%/yr.

Thermal resistance per m²: R(t) = 1/8 + 0.1/0.15 + t/0.025 + 1/20= 0.125 + 0.667 + 40t + 0.050 = 0.842 + 40t m²K/W

Heat gain rate: Q = 1200×56/(0.842+40t) = 67200/(0.842+40t) W

Annual electricity cost (8760 hr/yr):

$$E_{annual} = Q \times 8760/(COP \times 1000 \times 0.14)^{\{-1\}} \ldots properly:$$

$$E_{annual} = [Q\ W\ /\ COP\ ] \times 8760\ hr \times 0.14\ \$/kWh \times 10^{-3}\ kW/W$$

$$= 67200/(0.842 + 40t) \times (8760 \times 0.14)/(2.8 \times 1000)$$

$$= 67200/(0.842 + 40t) \times 0.4380 = 29{,}434/(0.842 + 40t)\ \$/yr$$

$$Present\ value\ of\ energy\ (annuity\ factor\ AF = \left[1 - 1.06^{\{-25\}}\right]/0.06 = 12.783):$$

$$PV_energy = 29{,}434 \times 12.783/(0.842 + 40t) = 376{,}329/(0.842 + 40t)$$

$$Capital\ cost\ of\ foam: C_{foam} = 85 \times 1200 \times t = 102{,}000t\ \$$$

$$Total\ cost: C(t) = 102{,}000t + 376{,}329/(0.842 + 40t)$$

$$Optimise: dC/dt = 102{,}000 - 376{,}329 \times 40/(0.842 + 40t)^2 = 0$$

$$(0.842 + 40t)^2 = 376{,}329 \times 40/102{,}000 = 15{,}053{,}160/102{,}000 = 147.6$$

$$0.842 + 40t = \sqrt{147.6} = 12.15 \rightarrow 40t = 11.31 \rightarrow t^* = 0.2827\ m \approx 283\ mm$$

$$d^2C/dt^2 = 2 \times 376{,}329 \times 40^2/(0.842 + 40t^*)^3 > 0\ \checkmark(minimum)$$

$$Total\ cost\ at\ t^*: PV_energy = 376{,}329/12.15 = \$30{,}972;\ C_foam = 102{,}000 \times 0.2827 = \$28{,}835$$

$$C_{total} = \$59{,}807$$

Sensitivity: $dC^* = /dt|_{\{t=t*\pm 50mm\}}: Small — near - optimum\ zone\ is\ flat\ (insensitive).$

$$At\ t = 233\ mm\ (50mm\ less): C = 102{,}000 \times 0.233 + 376{,}329/(0.842 + 9.32)$$
$$= \$23{,}766 + \$37{,}290 = \$61{,}056$$

$$Only\ \$1249\ more — confirms\ wide\ near - optimum\ zone.$$

$$Extension: (1)\ Repeat\ for\ electricity\ at\ \$0.25/kWh. (2)\ Find\ the\ payback\ period\ for\ t^*\ vs\ t = 100mm.$$

(3) How does optimal t change if COP drops to 2.0 (older equipment)?

✦ Design Problem 8.6.2 — Hydraulically Efficient Trapezoidal Channel

Trapezoidal channel: bottom width b, depth y, side slope 1H:1V (45°).

Cross-section: $A = by + y^2$ (given). Wetted perimeter: $P_w = b + 2\sqrt{2}\,y$.

Constraint: $A = A_0 = 1.8\,m^2$ (fixed flow area). Manning's $n = 0.014, S = 0.0005$.

Goal: Maximise $Q = (1/n)A\,R^{\{\frac{2}{3}\}}S^{\{\frac{1}{2}\}}\ where\ R = A/P_w$.

For fixed $A\ and\ S, maximise\ R \leftrightarrow minimise\ P_w$.

Express b from area constraint: $b = (A_0/y) - y$

Substitute into P_w: $P_w(y) = A_0/y - y + 2\sqrt{2}\,y = A_0/y + (2\sqrt{2}-1)y = 1.8/y + 1.828y$

Minimise: $dP_w/dy = -1.8/y^2 + 1.828 = 0$

$$y^2 = 1.8/1.828 = 0.9847 \rightarrow y^* = 0.9923\,m$$

$$d^2P_w/dy^2 = 3.6/y^3 > 0\ \checkmark(minimum)$$

b* = 1.8/0.9923 − 0.9923 = 1.8140 − 0.9923 = 0.8217 m

Check: A = 0.8217×0.9923 + (0.9923)² = 0.8151 + 0.9847 = 1.800 m² ✓

$$P_w^* = 1.8/0.9923 + 1.828 \times 0.9923 = 1.814 + 1.814 = 3.628\,m$$

$$R^* = A/P_w^* = 1.800/3.628 = 0.4963\,m$$

$$Q^* = \left(\frac{1}{0.014}\right)(1.800)(0.4963)^{\{\frac{2}{3}\}}(0.0005)^{\{\frac{1}{2}\}}$$

$$= 71.43 \times 1.800 \times 0.6226 \times 0.02236 = 1.788\,m^3/s$$

Compare with suboptimal y = 0.8 m (fixed arbitrarily):

$$b = \frac{1.8}{0.8} - 0.8 = 1.45\,m;\ P_w = \frac{1.8}{0.8} + 1.828 \times 0.8 = 2.25 + 1.462 = 3.712\,m$$

$$R = \frac{1.80}{3.712} = 0.4850;\ Q = 71.43 \times 1.800 \times 0.4850\text{^}\{0.667\} \times 0.02236 = 1.748\,m^3/s$$

Penalty for non-optimal design: (1.788−1.748)/1.788 = 2.2% flow reduction.

Extension: (1) Derive the result for a rectangle (side slopes = 0): show b = 2y.

(2) Show the optimal trapezoid is inscribed in a semicircle of radius y.

(3) Find the optimal rectangular channel for the same A0 and compare Q.

✦ Design Problem 8.6.3 — PID Controller: Derivative Term Analysis

Process: first-order thermal system with $\tau_p = 120\ s, K_p = 2°C/W$.

Step response: $T(t) = T_{set}\left(1 - e^{\left\{-\frac{t}{\tau_p}\right\}}\right)$ for unit step in heater power.

PID control law: $u(t) = K_c[e + (1/T_I)\int e\,dt + T_D\,de/dt]$

where $e = T_{set} - T\ (error)$. $Design$: $K_c = 2\ W/°C, T_D = 30\ s$.

$$At\ t = 60\ s\ after\ a\ 100°C\ setpoint\ step\ from\ T_0 = 0°C:$$

$$T(60) = 100\left(1 - e^{\left\{-\frac{60}{120}\right\}}\right) = 100\left(1 - e^{\{-0.5\}}\right) = 100 \times 0.3935 = 39.35°C$$

$$e(60) = 100 - 39.35 = 60.65°C$$

Rate of temperature rise: $\frac{dT}{dt} = (K_p/\tau_p) \times e^{\left\{-\frac{t}{\tau_p}\right\}} \times (heater\ power)$ …

More directly: in closed loop with proportional action only:

$\frac{dT}{dt}|_{\{t=60\}} \approx (T_{set} - T(60))/\tau_p \times K_p = 60.65 \times 2/120 = 1.011°C/s$ (open loop estimate)

$\frac{de}{dt} = -dT/dt = -1.011°C/s$ (error decreasing)

PID output components at t = 60 s:

$$u_P = K_c \times e = 2 \times 60.65 = 121.3\ W$$

$$u_D = K_c \times T_D \times de/dt = 2 \times 30 \times (-1.011) = -60.66\ W$$

$$u_{total}\ (P + D) = 121.3 - 60.7$$
$$= 60.6\ W\ (derivative\ term\ halves\ the\ control\ action!)$$

Physical interpretation:

Without D: full 121 W applied despite T rising rapidly → likely overshoot

With D: derivative 'sees' that T is rising fast and pre-emptively reduces heater power

Result: smoother approach to setpoint, reduced overshoot, less oscillation.

Noise sensitivity: differentiator amplifies noise with $gain = \omega T_D$

At f = 10 Hz (ω = 62.8 rad/s): gain = 62.8×30 = 1884× (extreme noise amplification)

Practical solution: filtered derivative $u_D = K_c\,T_D\,s/(1 + T_D\,s/N)\ with\ N = 10$

Maximum gain of filtered $D = K_c \times N = 2 \times 10 = 20\ W/°C\ (bounded)$

Extension: (1) Calculate the noise RMS amplification for $N = 5$ *vs* $N = 20$.

(2) Design a setpoint filter to avoid derivative kick on setpoint changes.

✦ Design Problem 8.6.4 — Minimum-Weight Fatigue-Constrained Tension Link

Fluctuating load: F(t) = 60,000 + 20,000 sin(ωt) N

Steel: $f_u = 900\ MPa, f_y = 750\ MPa, S_e = 360\ MPa.$

Safety factors: $n_f = 2.0\ (fatigue), n_y = 1.5\ (yielding).$

Stresses: $\sigma_{mean} = 60{,}000/A;\ \sigma_a = 20{,}000/A$

Goodman fatigue criterion: $\sigma_a/S_e + \sigma_{mean}/f_u = 1/n_f$

$$20{,}000/(360A) + 60{,}000/(900A) = 1/2$$

$$(55.56 + 66.67)/A = 0.500$$

$$A_{fat} = 122.22/0.500 = 244.4\ mm^2$$

$$Yield\ criterion: (\sigma_{mean} + \sigma_a)/f_y = 1/n_y$$

$$A_{yld} = n_y(F_{mean} + F_a)/f_y = 1.5 \times 80{,}000/750 = 160.0\ mm^2$$

Governing: A_{min} = max(244.4, 160.0) = 244.4 mm² (fatigue governs)

Minimum diameter: d = √(4A/π) = √(4×244.4/π) = √311.0 = 17.64 mm → use 18 mm

Sensitivity analysis — derivatives of required area:

$$dA/d(F_a) = n_f/S_e = 2.0/360 = 5.556 \times 10^{-3}\ mm^2/N$$

$$dA/d(F_{mean}) = n_f/f_u = 2.0/900 = 2.222 \times 10^{-3}\ mm^2/N$$

Ratio = 5.556/2.222 = 2.5 — alternating load is 2.5× more damaging than mean load.

This is a direct consequence of the Goodman line gradient.

Modified Goodman vs Gerber: Gerber uses $(\sigma_{mean}/f_u)^2$ instead of linear term.

Gerber $A_f at$: $\sigma_a/S_e + (\sigma_{mean}/f_u)^2 = 1/n_f$

20,000/(360A) + (60,000/(900A))² = 0.5 — nonlinear in A, solve numerically.

(Goodman is more conservative — standard design choice.)

Extension: (1) Find A_{min} if F_a increases to 35,000 N. (2) What S_e is needed for A = 200 mm²?

(3) How does the result change if f_u increases to 1200 MPa (higher-strength steel)?

✦ Design Problem 8.6.5 — Solar Collector Tilt Optimisation (Nairobi, 1.3°S)

Flat-plate collector at latitude $\varphi = 1.3°S$. $G_H = 850\ W/m^2$ (beam irradiance at noon).

For south-facing collector tilted at angle β from horizontal, solar declination δ:

Angle of incidence $\theta_i = |\varphi - \beta + \delta|$ (simplified, noon condition, φ in degrees).

Irradiance on tilted surface: $G_T = G_H\, cos(\theta_i)$.

Optimise for equinox (δ = 0):

$$G_T(\beta) = 850\, cos|1.3° - \beta|$$

$$dG_T/d\beta = 850\, sin(1.3° - \beta) \times sign(1.3° - \beta) = 0$$

$$\rightarrow \beta^* = 1.3°\ (collector\ tilted\ at\ latitude\ angle —\ equatorial\ rule)$$

$$G_T^* = 850\, cos(0) = 850\ W/m^2$$

$$\frac{d^2G_T}{d\beta^2} = -850\, cos(0) = -850 < 0\ \checkmark(maximum)$$

Seasonal variation:

June solstice $\delta = +23.45°$: $\theta_i = |1.3 - \beta + 23.45| = |24.75 - \beta|$; $\beta^* = 24.75°$

Dec solstice $\delta = -23.45°$: $\theta_i = |1.3 - \beta - 23.45| = |\beta + 22.15|$; $\beta^* = -22.15°$ (not practical)

Annual average: optimum $\beta^* \approx \varphi = 1.3°$

Sensitivity at equinox: $G_T(\beta) = 850\, cos(\beta - 1.3°)$

For β = 0 (horizontal): $G_T = 850\, cos(1.3°) = 850 \times 0.99974 = 849.8\ W/m^2$

Loss relative to $\beta^* = 1.3°$: $(850.0 - 849.8)/850 = 0.024\%$ — negligible!

$dG_T/d\beta|_{\{\beta=0\}} = -850\, sin(1.3°) = -850 \times 0.02269 = -19.3\ W/m^2$ per degree

$At\ \beta = 10°: loss = 850(1 - cos(8.7°)) = 850 \times 0.01147 = 9.75\ W/m^2$ (1.15% $loss$)

Conclusion: Nairobi's near-equatorial location means tilt angle is almost irrelevant for annual irradiance — a horizontal mount loses < 2% annually vs optimal tilt.

Economic recommendation: use horizontal mounting (simplest, lowest cost, no self-cleaning issues).

Extension: (1) Find optimal monthly β for Jan, Apr, Jul, Oct in Nairobi.

(2) Calculate annual energy using $E = \int_0^{8760} G_T(t)\,dt\ for\ \beta = 0\ vs\ \beta = 1.3°$.

(3) Extend to include diffuse radiation $G_d = 150\ W/m^2$ (isotropic model).

📝 Open-Ended Design Exercises 8.6

8.6.1 THERMAL: An LED module (48 W) needs an aluminium heat sink ($k = 205\ W/m \cdot K$, $h = 35\ W/m^2 \cdot K$). 12 rectangular fins, $t = 2\ mm, w = 25\ mm, pitch\ p = 8\ mm$.

Base must stay < 75°C (ambient 25°C). Find minimum fin height H.

Then minimise total heat sink mass by choosing H and the number of fins N.

8.6.2 KINEMATICS: A race car has thrust $F(v) = 12{,}000 - 80v^2\ N, m = 750\ kg$.

(a) $\frac{dv}{dt} = F(v)/m$ — $find\ v_T$ (b) $solve\ v(s)\ using\ v\,dv/ds = F(v)/m$

(c) $distance\ to\ reach\ 80\%\ of\ v_T$ (d) $jerk\ at\ t = 0$ (e) $critique\ model$.

8.6.3 ELECTRICAL: Design a Butterworth RC low-pass filter, $f_c = 3\ kHz$.

$v_{out}/v_{in} = 1/\sqrt{1 + \left(\frac{f}{f_c}\right)^2}$. (a) Find –3 dB cutoff formula

(b) select R, C (c) find $\frac{d(attenuation_{dB})}{df}$ beyond f_c

(d) for 40 dB/decade roll-off, how many identical stages are needed?

8.6.4 STRUCTURAL: A traffic sign post ($hollow\ steel\ tube, E = 200\ GPa$) carries a 1.2 m × 0.8 m sign at 5 m height. Design wind pressure = 1.2 kPa.

Deflection limit H/150. Find minimum I, select tube section, check buckling under axial weight ($sign + post = 350\ N$). Comment on governing failure mode.

8.6.5 FLUIDS: A pump delivers Q = 0.04 m³/s of water through a branching network:

Branch A: L = 400 m, required $Q_A = 0.025\ m^3/s$.

Branch B: L = 600 m, required $Q_B = 0.015\ m^3/s$.

Both branches have the same allowable head loss = 6 m.

$$Use\ \Delta P = \frac{8fL\rho Q^2}{\pi^2 D^5}\ with\ f = 0.020.\ Find\ D_A\ and\ D_B.$$

Apply $\frac{dD}{dQ}$ to assess sizing sensitivity.

Chapter 8 Summary and Key Results

This chapter has demonstrated that differential calculus is the unifying mathematical language of quantitative engineering. Across six distinct domains, the derivative appears in structurally analogous — though physically diverse — governing equations. The complete picture:

Section	***Physical Principle***	***Governing Equation***	***Key Result***
8.1 *Kinematics*	*Newton's 2nd Law*	$m\,\dot{v} = F(v,s,t)$	$v_T = \sqrt{(mg/k)};\ v(t) = v_T\ tanh(\frac{gt}{v_T})$
8.2 *Lumped Thermal*	*1st Law (closed system)*	$mc_p\,\theta = -hA\theta$	$T(t) = T_\infty + \theta_0\,(e^{\{-\frac{t}{\tau}\}});\ \tau = \frac{mc_p}{hA}$
8.2 *Fin Equation*	*Steady energy balance*	$\frac{d^2\theta}{dx^2} = m^2\theta$	$\frac{\theta}{\theta_b} = \frac{cosh[m(L-x)]}{cosh(mL)}$
8.3 *Draining Tank*	*Mass conservation*	$A_t\,ħ = -C_d\,A_o\sqrt{(2gh)}$	$h(t) = [\sqrt{h_0} - (C_dA_o/A_t)\sqrt{(g/2)}t]^2$
8.3 *Pipe Flow*	*Navier − Stokes (laminar)*	$\mu\nabla^2 v = \nabla p$	$Q = \pi R^4 \Delta P/(8\mu L)$
8.4 *RC Circuit*	*KVL* $+\ i = C\ dv/dt$	$\tau\ dv_C/dt + v_C = V_s$	$v_C = V_s(1 - e^{\{-\frac{t}{\tau}\}});\ \tau = RC$
8.4 *RLC Circuit*	*KVL (2nd order)*	$\ddot{y} + 2\alpha\dot{y} + {\omega_0}^2 y = 0$	$\zeta = \alpha/\omega_0;\ \omega_d = \omega_0\sqrt{(1-\zeta^2)}$
8.5 *Beam Bending*	*Flexure formula*	$\sigma = \frac{My}{I};\ \tau = \frac{VQ}{Ib}$	$\frac{d\sigma}{dy} = \frac{M}{I};\ \tau_max = \frac{3V}{2A}$
8.5 *Column Buckling*	*Euler BVP*	$EI\,y'' + Py = 0$	$P_cr = \pi^2 EI/L^2$
8.6 *Optimisation*	*Minimise cost/weight*	$\frac{dC}{dD} = 0$	$D^* = \left(\frac{5k_2}{k_1}\right)^{\{\frac{1}{6}\}};\ t^*_{ins} = analytical$

The six-step modelling methodology — (1) identify conservation principle, (2) apply to differential element, (3) take limit to obtain governing equation, (4) solve analytically, (5) interpret physically, (6) optimise — is the foundation of engineering science. Mastering this process, rather than memorising formulas, is the primary goal of this chapter.

The electrical-mechanical analogy (mass ↔ inductance, damping ↔ resistance, stiffness ↔ 1/capacitance) and the thermal-electrical analogy (temperature ↔ voltage, heat flow ↔ current, thermal resistance ↔ electrical resistance) demonstrate the deep mathematical unity underlying apparently different physical phenomena. Engineers who recognise these analogies can transfer solution techniques across domains — one of the most powerful skills in applied mathematics.

Chapter Review Problems

📝 Integrated Review Problems — Chapter 8

R7.1 A parachutist ($m = 90\ kg, k = 0.35\ kg/m$) opens the chute at $v_0 = 50\ m/s$.

(a) Find v_T (b) Set up and solve the ODE for v(t) with initial condition $v_0 \neq 0$

[Hint: use partial fractions differently from the zero-IC case] (c) Find t to reach

within 2% $of\ v_T$ (d) Compute jerk at t = 0 and at t = 10 s.

R7.2 An aluminium fin ($k = 200\ W/m \cdot K, 15 \times 2\ mm, L = 60\ mm, h = 60\ W/m^2 \cdot K$).

$T_b = 150°C, T_\infty = 20°C. Find: (a) m \frac{(b)\theta(x)}{\theta_b} (c) Q_f in\ (d) \eta_f.$

R7.3 A tank ($A_t = 4\ m^2, h_0 = 5\ m$) drains through $A_o = 0.005\ m^2$ ($C_d = 0.62$)

while water is pumped in at $Q_{in} = 0.01\ m^3/s$. Derive the ODE for h(t).

Find the equilibrium depth h_{eq}. Is the equilibrium stable? (Check d(ħ)/dh.)

R7.4 Ideal LC circuit: $L = 80\ mH, C = 50\ \mu F. Initial: v_C(0) = 50\ V, i(0) = 0$. Find,

$(a)\omega^0 \quad (b) v_C(t)\ and\ i(t) \quad (c)\ i_{max} \quad (d)\ show\ total\ energy = ½Cv_0^2\ always.$

$(e)\ sketch\ the\ phase\ portrait\ i\ vs\ v_C$..

R7.5 Crane column $W310 \times 129$ ($A = 16{,}500\ mm^2, I = 308 \times 10^6\ mm^4, c = 165\ mm$).

$P = 800\ kN, M = 120\ kN \cdot m, L_e = 6\ m, E = 200\ GPa$..

Find: $(a)\sigma_{axial} \quad (b)\sigma_{bending\ at\ extreme\ fibres} \quad (c)\sigma_{max\ combined}$

$(d) P_{cr} \quad (e) interaction\ ratio \frac{P}{P_{cr}} + \frac{M}{M_{cr}} \leq 1.0$..

R7.6 (Design) An injection moulding nozzle ($m = 0.12\,kg, c_p = 480\,J/kg\cdot K$, $A_{conv} = 0.025\,m^2, h = 18\,W/m^2\cdot K$) *has heater power* $\dot{Q}_{gen} = 450\,W$,
Also loses heat

by conduction $\dot{Q}_{cond} = k_{cond}\,A_{cond}\frac{\Delta T}{L} = 16\times 8\times 10^{-6}\times\frac{\Delta T}{0.1}.T_\infty = 22°C,.$

$target\ T_{ss} = 230°C..$

(a) Set up energy ODE with two loss terms (b) find T_{ss} (c) find τ

(d) is 230°C achievable? If not, what $\dot{Q}_{gen}$ is needed?

R7.7 Derive from first principles the Hagen-Poiseuille equation $Q = \frac{\pi R^4 \Delta P}{8\mu L}$.

Show each integration step and the boundary conditions applied.

Verify dimensional homogeneity of the final result.

R7.8 For a series RLC circuit (R = 200 Ω, L = 100 mH, C = 10 μF), find:

(a) ω_0, α, ζ (b) ω_d if underdamped (c) $v_C(t)\ for\ v_C(0) = 200V, i(0) = 0$

(d) time and value of first peak (e) energy dissipated in first cycle.

Selected Answers

Review R7.1 — Parachutist with non-zero initial velocity

With vo ≠ 0 and quadratic drag, the ODE m dv/dt = mg − kv² separates as before, but integration constants differ. Using partial fractions:

$$\int dv/(v_T^2 - v^2) = (1/2v_T)[ln|(v_T + v)/(v_T - v)|] = gt/v_T^2 + C$$

$$At\ t = 0: \left(\frac{1}{2v_T}\right) ln\left|\frac{v_T + v_0}{v_T - v_0}\right| = C$$

$$\boldsymbol{v(t) = v_T \cdot \frac{\left[v_T\, tanh\left(\frac{gt}{v_T} + atanh\left(\frac{v_0}{v_T}\right)\right)\right]}{v_T} = v_T\, tanh(\frac{gt}{v_T} + \varphi_0)}$$

where $\varphi^0 = atanh\left(\frac{v^0}{v_T}\right) = atanh\left(\frac{50}{v_T}\right).With\ v_T = \sqrt{90\times\frac{9.81}{0.35}} = \sqrt{2523} = 50.23\frac{m}{s}$: $\varphi_0 = atanh(50/50.23) = atanh(0.9954) = 3.027\,rad$

$$v(t) = 50.23\,tanh(9.81t/50.23 + 3.027)t\ to\ within\ 2\%\ of\ v_T: tanh(0.1953t + 3.027) = 0.98$$

$$0.1953t + 3.027 = atanh(0.98) = 2.298\ldots \rightarrow\ t\ is\ negative \rightarrow\ already\ past\ v_T\ from\ the\ start!\ Since\ v_0 = 50\ m/s \approx v_T = 50.23\ m/s,$$ parachutist is almost immediately at terminal velocity

Review R7.4 — Ideal LC Circuit

$$\omega_0 = 1/\sqrt{(LC)} = 1/\sqrt{(0.08 \times 50 \times 10^{-6})} = 1/\sqrt{(4 \times 10^{-6})} = 500\ rad/s$$

$$v_C(t) = 50\ cos(500t)V\ i(t) = -C\frac{dv_C}{dt} = 50 \times 10^{-6} \times 500 \times 50\ sin(500t) = 1.25\ sin(500t)\ A$$

$$i_{max} = \frac{v_C(0)}{\sqrt{\frac{L}{C}}} = \frac{50}{\sqrt{\frac{0.08}{50 \times 10^{-6}}}} = \frac{50}{\sqrt{1600}} = \frac{50}{40} = 1.25\ A\ \checkmark$$

$$E_{total} = ½Cv_C^2 + ½Li^2 = ½ \times 10^{-4} \times 2500cos^2 + ½ \times 0.08 \times 1.5625sin^2(500t)$$

$$= 0.125cos^2(500t) + 0.0625sin^2(500t)\ldots = 0.125\ J\ always\ (const)\ \checkmark [using\ ½Cv_0{}^2 = ½ \times 10^{-4} \times 2500 = 0.125J]$$

Review R7.7 — Hagen-Poiseuille Derivation Checklist

Starting from the axisymmetric Navier-Stokes equation (fully developed, steady, incompressible):

$$0 = -\frac{\partial p}{\partial x} + \mu\left[\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right] \rightarrow \frac{1}{r} \cdot \frac{d}{dr}(r\frac{dv}{dr}) = (\frac{\frac{1}{\mu}dp}{dx} = -\frac{\Delta P}{\mu L}$$

$$Step\ 1: Multiply\ by\ r: \frac{d}{dr}(r\frac{dv}{dr}) = -(\frac{\Delta P}{\mu L})r$$

$$Step\ 2: Integrate: r\frac{dv}{dr} = -\left(\frac{\Delta P}{2\mu L}\right)r^2 + C^1;\ at\ r = 0: \frac{dv}{dr} = 0 \rightarrow C_1 = 0$$

$$Step\ 3: \frac{dv}{dr} = -(\frac{\Delta P}{2\mu L})r$$

$$Step\ 4: Integrate: v = -(\frac{\Delta P}{4\mu L})r^2 + C_2;\ at\ r = R: v = 0 \rightarrow C_2 = \Delta PR^2/(4\mu L)$$

$$Step\ 5: v(r) = \frac{\Delta P R^2}{4\mu L} \times \left[1 - \left(\frac{r}{R}\right)^2\right];\ v_{max} = \frac{\Delta P R^2}{4\mu L}$$

$$Step\ 6: Q = \int_o^R v(r) \times 2\pi r\, dr = 2\pi \frac{\Delta P R^2}{4\mu L} \int_0^R \left(1 - \frac{r^2}{R^2}\right) r\, dr$$

$$= 2\pi \frac{\Delta P R^2}{4\mu L} \times \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 2\pi \frac{\Delta P R^2}{4\mu L} \times \frac{R^2}{4} = \frac{\pi R^4 \Delta P}{8\mu L}\ \checkmark$$

$$Dimensional\ check: [Q] = \frac{[m^4][Pa]}{[Pa \cdot s][m]} = m^4 \cdot \frac{\frac{N}{m^2}}{N \cdot \frac{s}{m^2} \cdot m} = \frac{m^4}{s \cdot m} \times \frac{m}{m} = m^3/s\ \checkmark$$

CHAPTER 8

Engineering Modeling and Integrated Problems

Kinematics • Thermal Systems • Fluid Dynamics • Electrical Circuits • Structural Analysis

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Chapter Introduction

Engineering analysis is, at its core, an exercise in mathematical modeling. Every physical system — a cooling fin, a vibrating structure, a charging capacitor, a draining tank, a loaded column — obeys fundamental conservation laws that, when expressed in the language of mathematics, produce relationships involving rates of change. Differential calculus is therefore not merely a tool of abstract mathematics; it is the foundational language through which engineers describe, predict, and control the behaviour of physical systems.

This chapter bridges the gap between pure calculus theory and the rich diversity of engineering practice. Rather than presenting calculus as a set of isolated techniques, we show how the derivative, in its various interpretations and notations, enters naturally and unavoidably into the governing equations of six major engineering domains. Each section follows a consistent modelling framework: (1) state the physical principle, (2) apply it to a differential element, (3) take appropriate limits to obtain a governing differential or algebraic equation, (4) solve analytically, and (5) interpret every result in physical and engineering terms.

Throughout the chapter, authentic engineering symbols are used: θ for temperature excess, ρ for density, μ for dynamic viscosity, ω for angular frequency, σ for normal stress, τ for shear stress or time constant, ζ for damping ratio, α for thermal diffusivity or damping coefficient, and β for thermal expansion coefficient or tilt angle. Mathematical operators are written in full symbolic notation: ∂, $\int$, $\surd$, Σ, Δ, and the prime/dot conventions for time and spatial derivatives.

All worked examples use real engineering data drawn from civil, mechanical, electrical, and chemical engineering practice. Every numerical result has been independently verified, and the derivations proceed from first principles using the techniques developed in earlier chapters.

8.1 Kinematics Modeling

8.1.1 The Kinematic Derivative Chain

Kinematics describes the geometry of motion independently of the forces that cause it. For a particle moving along a curved path in space, the position vector r(t) = [x(t), y(t), z(t)] fully characterises the motion. All kinematic quantities follow by successive differentiation:

$$\boldsymbol{v(t)} = \dot{\mathrm{r}} = \frac{\boldsymbol{dr}}{\boldsymbol{dt}}\ [\boldsymbol{velocity} — \boldsymbol{m/s}]$$

$$\boldsymbol{a(t)} = \dot{\boldsymbol{v}} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \ddot{\boldsymbol{r}}\ [\boldsymbol{acceleration} — \boldsymbol{m/s^2}]$$

$$\boldsymbol{j(t)} = \dot{\mathrm{a}} = \frac{\boldsymbol{da}}{\boldsymbol{dt}} = \dddot{\boldsymbol{r}}\ [\boldsymbol{jerk} — \boldsymbol{m/s^3}]$$

For one-dimensional rectilinear motion the magnitudes are scalars $s(t), v(t) = \dot{s}, a(t) = \dot{v} = \ddot{s}$. An extremely useful identity derived from the chain rule allows velocity to be expressed as a function of position rather than time:

$$\boldsymbol{a} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \left(\frac{\boldsymbol{dv}}{\boldsymbol{ds}}\right)\left(\frac{\boldsymbol{ds}}{\boldsymbol{dt}}\right) = \boldsymbol{v} \cdot \frac{\boldsymbol{dv}}{\boldsymbol{ds}}$$

This transformation converts a differential equation in (v, t) into one in (v, s), enabling closed-form integration whenever the force — and hence a — is given as a function of position s rather than time t. The chain rule identity v dv = a ds integrates directly to give the work-energy theorem:

$$\int \boldsymbol{v}\,\boldsymbol{dv} = \int \boldsymbol{a}\,\boldsymbol{ds} \implies \tfrac{1}{2}\boldsymbol{v}^2 - \tfrac{1}{2}\boldsymbol{v_0}^2 = \int_{\{\boldsymbol{s_0}\}}^{\{\boldsymbol{s}\}} \boldsymbol{a}\,\boldsymbol{ds}$$

In two dimensions, the position vector components x(t) and y(t) obey independent second-order equations from Newton's second law:

$$\boldsymbol{m}\,\ddot{\mathrm{x}} = \boldsymbol{F}_{\mathrm{x}}(\boldsymbol{x}, \boldsymbol{y}, \dot{\mathrm{x}}, \dot{\mathrm{y}}, \boldsymbol{t})\ \boldsymbol{m}\,\ddot{\mathrm{y}} = \boldsymbol{F_y}(\boldsymbol{x}, \boldsymbol{y}, \dot{\mathrm{x}}, \dot{\mathrm{y}}, \boldsymbol{t})$$

When the force components are independent (as in standard projectile motion without cross-coupling), the two equations may be solved separately — the power of separation in kinematic analysis.

8.1.2 Variable Acceleration — Newton's Second Law

The most general kinematic problem involves a force that depends simultaneously on time t, position s, and velocity v. Newton's second law provides:

$$\boldsymbol{m \cdot \frac{dv}{dt} = F(s, v, t)}$$

Special cases and their solution strategies are summarised below:

Force Type	***Governing Equation***	***Solution Strategy***
$F = F(t)\ only$	$m\frac{dv}{dt} = F(t)$	$Direct\ integration: v(t) = (1/m)\int F(t)dt$
$F = F(v)\ only$	$m\frac{dv}{dt} = F(v)$	$Separable: m\frac{dv}{F(v)} = dt, integrate\ both\ sides$
$F = F(s)\ only$	$mv\frac{dv}{ds} = F(s)$	$Use\ v\ dv = \left(\frac{F}{m}\right)ds, integrate: ½v^2 = \left(\frac{1}{m}\right)\int F\ ds$
$F = F(v) — position$	$mv\frac{dv}{ds} = F(v)$	$Separable\ in\ v, s: v\frac{dv}{F(v)} = ds/m$

The most common engineering case with pure velocity dependence is aerodynamic drag. Two drag models are used in practice: linear drag $F_d = -bv$ (Stokes drag, valid for low Reynolds number Re < 1, e.g., small droplets) and quadratic drag $F_d = -kv^2$ (Newtonian drag, valid for Re > 1000, e.g., vehicles, aircraft, projectiles).

8.1.3 Quadratic Air Resistance and Terminal Velocity — Full Derivation

Consider a body of mass m falling vertically from rest under gravity g with quadratic drag. Taking downward as positive:

$$\boldsymbol{m \cdot dv/dt = mg - kv^2}$$

where $k = ½\, C_D\, \rho_{air}$ A is the aerodynamic drag constant [kg/m], C_D is the drag coefficient (dimensionless), ρ_{air} is air density [kg/m³], and A is the body's frontal area [m²]. Setting dv/dt = 0 to find the terminal velocity v_T :

$$\boldsymbol{v_T = \sqrt{(mg/k)}}$$

Rewriting the equation of motion in terms of v_T :

$$dv/dt = g(1 - v^2/v_T^2)$$

Separating variables:

$$dv/(1 - v^2/v_T^2) = g\ dt$$

Using partial fractions: $1/(1 - u^2) = ½[1/(1-u) + 1/(1+u)], with\ u = v/v_T$:

$$(v_T/2)[ln(1 + v/v_T) - ln(1 - v/v_T)] = gt + C$$

With $v(0) = 0$: $C = 0$. *Noting that* $(1/2)ln[(1+u)/(1-u)] = atanh(u)$:

$$\boldsymbol{v(t) = v_T \cdot tanh(gt/v_T)}$$

Integrating once more for position s(t), with s(0) = 0:

$$s(t) = \int_0^t v_T \ tanh(g\tau/v_T)\, d\tau = v_T \cdot [v_T/g \cdot ln\, cosh(g\tau/v_T)]_0^t$$

$$\boldsymbol{s(t) = (v_T^2/g) \cdot ln[cosh(gt/v_T)]}$$

Key derivative properties: at t = 0, v = 0 and a = g (full gravitational acceleration). As $t \to \infty, tanh \to 1, so\ v \to v_T\ and\ a \to 0$. The time to reach fraction f of terminal velocity is:

$$\boldsymbol{t_f = (v_T/g) \cdot atanh(f)}$$

8.1.4 Projectile Motion with Linear Drag

For a projectile launched at speed v0 and elevation angle α above horizontal, with linear drag coefficient b [kg/s], Newton's second law resolves into:

$$\boldsymbol{m}\,\ddot{x} = -\boldsymbol{b}\dot{x}\ \boldsymbol{m}\,\ddot{y} = -\boldsymbol{mg} - \boldsymbol{b}\dot{y}$$

Defining the drag time constant $\tau = m/b\ [s]$, the horizontal equation becomes $\ddot{x} + \dot{x}/\tau = 0$, a first-order ODE in $\dot{x}$. Integrating:

$$\dot{x}(t) = v_0 cos\alpha \cdot e^{-\frac{t}{\tau}}$$

$$x(t) = v_0 \tau\, cos\alpha \cdot (1 - e^{-\frac{t}{\tau}})$$

The vertical equation $\ddot{y} + \dot{y}/\tau = -g$ is a first-order linear ODE. Multiplying by the integrating factor $e^{\frac{t}{\tau}}$:

$$d/dt[\dot{y}\, e^{\frac{t}{\tau}}] = -g\, e^{\frac{t}{\tau}}$$

$$\dot{y}\, e^{\frac{t}{\tau}} = -g\tau\, e^{\frac{t}{\tau}} + C$$

With $\dot{y}(0) = v_0 sin\alpha$: $C = v_0 sin\alpha + g\tau$. Therefore:

$$\dot{y}(\boldsymbol{t}) = -\boldsymbol{g\tau} + (\boldsymbol{v^0 sin\alpha} + \boldsymbol{g\tau})\boldsymbol{e^{-\frac{t}{\tau}}}$$

$$\boldsymbol{y(t) = -g\tau t + \tau(v_0 sin\alpha + g\tau)(1 - e^{-\frac{t}{\tau}})}$$

Note that as $t \to \infty, \dot{x} \to 0\ and\ \dot{y} \to -g\tau = -mg/b$ (terminal fall velocity). The horizontal range at landing (y = 0) must be found numerically in general, except for special cases.

Rate of change of range with launch angle — an optimisation of practical importance in ballistics:

Setting $\partial R/\partial \alpha = 0$ gives the optimal angle α*, which for linear drag is always less than 45°.

In the limit b → 0 (τ → ∞), using the Taylor expansion $e^{-\frac{t}{\tau}} \approx 1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \ldots$, both x(t) and y(t) recover the drag-free results: $x = v_0 cos\alpha \cdot t \; and \; y = v_0 sin\alpha \cdot t - ½gt^2$. This is an important consistency check.

8.1.5 Circular and Curvilinear Motion — Tangential and Normal Acceleration

For a particle moving along a curved path of radius of curvature ρ (not to be confused with density), the acceleration vector has two components:

$\boldsymbol{a_t = dv/dt = \dot{v}}$ **[tangential — along the path]**

$\boldsymbol{a_n = v^2/\rho}$ **[normal — directed toward centre of curvature]**

$\boldsymbol{|a| = \sqrt{(a_t^2 + a_n^2)}}$**.**

For circular motion of radius r (constant ρ = r), expressing position in terms of angular displacement $\theta(t)$: $s = r\theta, so \; v = r\theta = r\omega$, where ω = θ̇ is angular velocity [rad/s]. Differentiating again: $a_t = r\alpha, where \; \alpha = \dot{\theta} = \dot{\omega}$ is angular acceleration [rad/s²].

The normal (centripetal) acceleration: $a_n = v^2/r = r\omega^2$. For uniform circular motion $(\omega = const), a_t = 0 \; and \; a_n = r\omega^2$ — the acceleration is purely centripetal.

Newton's second law applied to rotating machinery:

$\boldsymbol{I \cdot \alpha = I \cdot d\omega/dt = M_}$**net [rotational analogue of F = ma]**

where I is the mass moment of inertia [kg·m²] and M_{net} is the net torque [N·m]. If $M_{net} = M_{drive} - M_{load} - c_r \, \omega \; (with \; rotational \; drag \; c_r \, \omega)$, the equation becomes:

$$\boldsymbol{I \cdot \frac{d\omega}{dt} + c_r \, \omega = M_{drive} - M_load}$$

This is a first-order linear ODE with the same structure as the RL circuit equation — a profound analogy that allows electrical engineers and mechanical engineers to share solution techniques.

✦ Worked Example 8.1.1 — Skydiver: Terminal Velocity and Fall Distance

A skydiver (m = 80 kg) jumps from rest. Drag constant k = 0.25 kg/m, g = 9.81 m/s².

(a) $v_T = \sqrt{(mg/k)} = \sqrt{(80 \times 9.81 \; / \; 0.25)} = \sqrt{3139.2} = 56.03 \; m/s$ (201.7 km/h)

(b) $v(10\,s) = 56.03 \times tanh(9.81 \times 10 / 56.03) = 56.03 \times tanh(1.7509)$

$tanh(1.7509) = (e^{3.502} - 1)/(e\text{^}3.502 + 1) = (33.18 - 1)/(33.18 + 1) = 0.9417$

$$v(10) = 56.03 \times 0.9417 = 52.77\,m/s$$

(c) $s(10) = (v_T^2/g)\,ln[cosh(gt/v_T)] = (3139.4/9.81)\,ln[cosh(1.7509)]$

$$cosh(1.7509) = \frac{\left(e^{1.7509} + e^{\{-1.7509\}}\right)}{2} = (5.761 + 0.1736)/2 = 2.967$$

$$s(10) = 320.0 \times ln(2.967) = 320.0 \times 1.0878 = 348.1\,m$$

Free-fall (no drag): s = ½gt² = 490.5 m → drag reduces fall by 29%

(d) Time to 95% of v_T: $t = (v_T/g)\,atanh(0.95) = (56.03/9.81) \times 1.8318 = 10.47\,s$

(e) Instantaneous heat rate at $t = 5\,s$: $Q = kv^2 = 0.25 \times v(5)^2$

$$v(5) = 56.03\,tanh(0.8745) = 56.03 \times 0.7036 = 39.43\,m/s$$

$$Q = 0.25 \times 1554.7 = 388.7\,W\ (power\ dissipated\ against\ drag)$$

✦ Worked Example 8.1.2 — Rocket with Decaying Thrust

Rocket: $m = 5\,kg, thrust\ F(t) = 200e^{\{-0.5t\}}\,N. Initial: s(0) = 0, v(0) = 0.$

Equation of motion: $m\,\dot{v} = 200e^{\{-0.5t\}} \rightarrow \dot{v} = 40e^{\{-0.5t\}}$

(a) $v(t) = \int 40\,e^{\{-0.5t\}}\,dt = -80e^{\{-0.5t\}} + C$

$$v(0) = 0: C = 80 \rightarrow v(t) = 80\left(1 - e^{\{-0.5t\}}\right)\,m/s$$

(b) $s(t) = \int 80\left(1 - e^{\{-0.5t\}}\right)dt = 80t + 160e^{\{-0.5t\}} + C_2$

$$s(0) = 0: C_2 = -160 \rightarrow s(t) = 80t - 160\left(1 - e^{\{-0.5t\}}\right)m$$

(c) $v_\infty = lim_{\{t\to\infty\}}\,80\left(1 - e^{\{-0.5t\}}\right) = 80\,m/s = 288\,km/h$

(d) Acceleration: $a(t) = \dot{v} = 40e^{\{-0.5t\}}\,m/s^2$

$$a(0) = 40\,m/s^2 = 4.08g\ (initial\ acceleration\ 4g)$$

$$a(4) = 40e^{\{-2\}} = 40 \times 0.1353 = 5.41\,m/s^2$$

(e) Jerk: $j(t) = \dot{a}(t) = -20e^{\{-0.5t\}}\,m/s^3$

$j(0) = -20\,m/s^3$ (acceleration decaying at 20 m/s³ — smooth profile)

Chapter 8 —

Applied Mathematical Modelling

8.1 Kinematics Modelling

Worked Example 8.1.3 — Spring Launcher: Velocity from Position

Spring-loaded launcher: k_s = 8 000 N/m, compression x_0 = 0.15 m, m = 2 kg. The energy method equates stored spring potential energy to kinetic energy at launch:

$$\frac{1}{2}mv^2_{\,L} = \frac{1}{2}k_s x^2_{\,0}$$

Solving for launch velocity:

$$v_L = x_0\sqrt{\frac{k_s}{m}} = 0.15 \times \sqrt{8\,000 / 2} = 0.15 \times 63.25 = 9.487 \text{ m/s}$$

Verification via the product form v dv = (k_s/m) x dx, integrating from 0 to x_0:

$$\frac{{v_L}^2}{2} = \frac{k_s}{2m}{x_0}^2 = \frac{8\,000 \times 0.0225}{2} = 45 \implies v_L = \sqrt{90} = 9.487 \text{ m/s } ✓$$

Maximum height for vertical launch:

$$H = \frac{{v_L}^2}{2g} = \frac{90}{2 \times 9.81} = 4.587 \text{ m}$$

Time to reach peak height:

$$t_{peak} = \frac{v_L}{g} = \frac{9.487}{9.81} = 0.9670 \text{ s}$$

Worked Example 8.1.4 — Rotating Drum: Angular Deceleration

Rotating drum: I = 12 kg·m², ω_0 = 150 rad/s. Friction torque M_U = (8 + 0.04ω) N·m.

(a) Equation of motion:

$$I\frac{d\omega}{dt} = -M_U = -(8 + 0.04\omega)$$

$$12\frac{d\omega}{dt} + \frac{\omega}{300} = -\frac{2}{3}$$

(b) Solution using integrating factor e^{t/300}:

$$\omega(t) = 350e^{-t/300} - 200 \text{ rad/s}$$

(c) Time to stop (ω = 0):

$$350e^{-t/300} = 200 \implies t_{stop} = -300\ln\left(\frac{4}{7}\right) = 167.9 \text{ s} \approx 2 \text{ min } 48 \text{ s}$$

(d) Total angle turned (integrate ω from 0 to 167.9 s):

$$\theta = \int_0^{167.9} \omega\, dt = \left[-105\,000e^{-t/300} - 200t\right]_0^{167.9} = 11\,420 \text{ rad}$$

$$\text{Revolutions} = \frac{11\,420}{2\pi} = 1\,817$$

Exercises 8.1 — Kinematics Modelling

8.1.1 A vehicle (m = 1 200 kg, k = 0.45 N·s²/m²) has initial speed v_0 = 30 m/s (engine off). Using $v\,dv/ds = -kv^2/m$, find v(s). At what distance does speed fall to 10 m/s?

8.1.2 A ball (m = 0.15 kg, b = 0.012 kg/s) is launched at v_0 = 40 m/s, α = 35°. Find x(t) and y(t) with linear drag (τ = m/b). Find the time to reach peak height.

8.1.3 A particle has $a(t) = 6\sin(2t) - 4\cos(t)$ m/s². Given s(0) = 2 m, v(0) = −1 m/s, find s(t) and evaluate s(π).

8.1.4 A flywheel (I = 25 kg·m²) is driven by torque M(t) = (120 − 0.5ω) N·m. With ω(0) = 0, find: (a) ω(t) (b) steady-state ω_{ss} (c) time to reach 90 % of ω_{ss}.

8.1.5 Derive $R = v_0^2\sin(2\alpha)/g$. Find dR/dα and confirm the optimal angle is α* = 45°. What is $d^2R/d\alpha^2$ at α = 45°?

8.1.6 A motor vehicle has thrust F_e = 3 000 N, rolling resistance $F_r = 50v + 0.4v^2$ N, m = 900 kg. Set up and solve for terminal speed.

8.2 Thermal Systems Modelling

8.2.1 Newton's Law of Cooling — Full Derivation

A solid body of mass m, specific heat c_p [J/kg·K], uniform temperature T(t), is immersed in fluid at ambient temperature T∞. Convection over surface area A with coefficient h [W/m²·K]. First law (no work, no internal generation):

$$mc_p \cdot \frac{dT}{dt} = -hA(T - T_\infty)$$

Define excess temperature $\theta(t) = T(t) - T\infty$. Then $d\theta/dt = dT/dt$ and:

$$\frac{d\theta}{dt} = -\frac{hA}{mc_p}\theta = -\frac{\theta}{\tau} \quad \text{where } \tau = \frac{mc_p}{hA} \text{ [s]}$$

Solution:

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-t/\tau}$$

Total heat transferred from t = 0 to t:

$$Q(t) = mc_p\theta_0\left(1 - e^{-t/\tau}\right) \text{ [J]}$$

Validity: Biot number $Bi = hL_C/k < 0.1$ (lumped-capacitance model). For $Bi > 0.1$ the full heat equation $\partial T/\partial t = \alpha\nabla^2 T$ must be solved, where $\alpha = k/(\rho c_p)$ is the thermal diffusivity [m^2/s].

8.2.2 Multi-Layer Thermal Resistance Networks

For a composite wall of n layers with convection on both sides, total thermal resistance:

$$R_{total} = \frac{1}{h_i A} + \Sigma_{j=1}^{n}\frac{L_j}{k_j A} + \frac{1}{h_e A} \text{ [K/W]}$$

Steady-state heat flux through the composite wall:

$$q = \frac{T_i - T_o}{\frac{1}{h_i} + \Sigma\ \frac{L_j}{k_j} + \frac{1}{h_e}} \text{ [W/m}^2\text{]}$$

Overall heat transfer coefficient U [$W/m^2 \cdot K$], defined by $q = U(T^I - T_o)$:

$$\frac{1}{U} = \frac{1}{h_i} + \Sigma\ \frac{L_j}{k_j} + \frac{1}{h_e}$$

8.2.3 The Extended Surface (Fin) Equation — Derivation

A straight fin of constant cross-section A_C, perimeter P, length L, conductivity k, extends from base temperature T^b. Energy balance on a differential slice dx (steady state):

$$\frac{d\theta^2}{dx^2} - m^2\theta = 0 \quad \text{where } m = \sqrt{\frac{hP}{kA_C}} \text{ [m}^{-1}\text{]}$$

General solution: $\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx)$

For an insulated tip ($d\theta/dx = 0$ at $x = L$), applying boundary conditions:

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh[m(L - x)]}{\cosh(mL)}$$

Heat dissipated by the fin:

$$Q_{\text{fin}} = \sqrt{hPkAc} \cdot \theta_b \cdot \tanh(mL) \ \text{[W]}$$

Fin efficiency (actual / maximum if fin were at T_b throughout):

$$\eta_f = \frac{\tanh(mL)}{mL}$$

Note: $\eta_U \to 1$ as $mL \to 0$ (high conductivity / short fin); $\eta_U \to 0$ as $mL \to \infty$ (long fin loses effectiveness).

8.2.4 Transient Heating with Internal Heat Generation

When a body generates heat internally at rate Q^G_{en} [W] (e.g. resistance heater, battery cell during fast charge), the lumped energy balance becomes:

$$mc_p \cdot \frac{dT}{dt} = Q_{\text{gen}} - hA(T - T_\infty)$$

Let $\theta = T - T\infty$ and $\theta\infty = Q^G_{en}/(hA)$ (steady-state excess temperature). Solution:

$$T(t) = T_\infty + \frac{Q_{\text{gen}}}{hA} + (T_0 - T_{ss})e^{-t/\tau}$$

where $T_{ss} = T\infty + Q^G_{en}/(hA)$. Rate of temperature rise at t = 0:

$$\left.\frac{dT}{dt}\right|_{t=0} = \frac{Q_{\text{gen}} - hA(T_0 - T_\infty)}{mc_p} \ [°C/s]$$

✦ Worked Example 8.2.1 — Steel Engine Cooling Fin (Insulated Tip)

k = 50 W/m·K, L = 80 mm, w = 25 mm, t = 4 mm, h = 85 W/m²·K, T_b = 180 °C, T∞ = 25 °C ⟹ θ_b = 155 °C.
Ac = 25 × 4 = 100 mm² = 1.0 × 10^{-4} m²; P = 2(25+4) = 58 mm = 0.058 m

(a) Fin parameter m:

$$m = \sqrt{\frac{hP}{kAc}} = \sqrt{\frac{85 \times 0.058}{50 \times 1.0 \times 10^{-4}}} = \sqrt{986} = 31.40 \ \text{m}^{-1}; \quad mL = 31.40 \times 0.080 = 2.512$$

(b) Tip temperature (x = L):

$$\frac{\theta(L)}{\theta_b} = \frac{1}{\cosh(mL)} = \frac{1}{\cosh(2.512)} = \frac{1}{6.205} = 0.1612 \implies T(L) = 25 + 0.1612 \times 155 = 50.0 \ °C$$

(c) Temperature at x = 40 mm:

$$\frac{\theta(0.04)}{\theta_b} = \frac{\cosh(1.256)}{\cosh(2.512)} = \frac{1.898}{6.205} = 0.3059 \implies \text{T}(40\text{ mm}) = 72.4\ ^\circ\text{C}$$

(d) Heat dissipated:

$$Q_{\text{fin}} = \sqrt{hPkAc}\cdot\theta_b\cdot\tanh(mL) = 0.1570 \times 155 \times 0.9870 = 24.02\text{ W}$$

(e) Fin efficiency:

$$\eta_f = \frac{\tanh(2.512)}{2.512} = \frac{0.9870}{2.512} = 0.3927 \approx 39.3\ \%\ \checkmark$$

(f) Temperature gradient at x = 30 mm:

$$\frac{d\theta}{dx}_{\text{x=0.03}} = -\theta_b\cdot m\cdot\frac{\sinh[\text{m(L}-\text{x})]}{\cosh(\text{mL})} = -155 \times 31.40 \times \frac{2.301}{6.205} = -1\,803\ ^\circ\text{C/m}$$

✦ Worked Example 8.2.2 — Lumped Capacitance: Battery Cell Heating

Li-ion cell: m = 0.048 kg, c_p = 1 000 J/kg·K, A = 0.008 m², h = 12 W/m²·K. Internal resistance R_{int} = 0.05 Ω; discharge current I = 20 A. $Q_{gen} = I^2R_{int} = 400 \times 0.05 = 20$ W. $T_0 = T_\infty = 25\ ^\circ$C.

(a) Thermal time constant:

$$\tau = \frac{mc_p}{hA} = \frac{0.048 \times 1\,000}{12 \times 0.008} = 500\text{ s}$$

(b) Steady-state temperature:

$$T_{ss} = T_\infty + \frac{Q_{gen}}{hA} = 25 + \frac{20}{12 \times 0.008} = 233.3\ ^\circ\text{C}$$

DANGER: T_{ss} > 80 °C (maximum for Li-ion) ⟹ thermal runaway risk!

(c) Time to reach T = 60 °C (warning threshold):

$$T(t) = 233.3 - 208.3e^{-t/500}\ ^\circ\text{C}$$

$$60 = 233.3 - 208.3e^{-t/500} \implies t = -500\ln\left(\frac{173.3}{208.3}\right) = 91.9\text{ s} \approx 1.5\text{ min}$$

(d) Initial rate of temperature rise:

$$\frac{dT}{dt}_{t=0} = \frac{Q_{gen}}{mc_p} = \frac{20}{0.048 \times 1\,000} = 0.4167\ ^\circ\text{C/s}\ (\approx 25\ ^\circ\text{C/min} —\ \text{extremely dangerous})$$

(e) Required h to keep $T_{ss} \le 60$ °C:

$$h = \frac{Q_{gen}}{(60-25) \times 0.008} = \frac{20}{35 \times 0.008} = 71.4\ \text{W/m}^2\cdot\text{K}$$

(Requires forced liquid cooling — natural convection h ≈ 10–25 is insufficient.)

✦ Worked Example 8.2.3 — Composite Wall: Cold Store Panel

Layers (inside to outside): h^{I} = 8 W/m²K | 12 mm plywood k = 0.15 | 150 mm polyurethane foam k = 0.025 | 2 mm steel k = 50 | h_e = 20 W/m²K. T^{I} = −18 °C, T_o = 38 °C, A = 1 m².

Total thermal resistance:

$$R_{\text{total}} = \frac{1}{8} + \frac{0.012}{0.15} + \frac{0.150}{0.025} + \frac{0.002}{50} + \frac{1}{20} = 6.255\ \text{m}^2\text{K/W}$$

Heat flux:

$$q = \frac{\Delta T}{R_{\text{total}}} = \frac{56}{6.255} = 8.953\ \text{W/m}^2$$

Interface temperatures (inside to outside): −18 → −16.88 → −16.16 → +37.55 → 37.55 → 38.00 °C. The foam carries ΔT_{foam} = 53.72 °C (96 % of total ΔT). Sensitivity:

$$\frac{\partial q}{\partial L_{\text{foam}}} = -\frac{q^2}{k_{\text{foam}}} = -\frac{(8.953)^2}{0.025} = -3\,208\ \text{W/m}^3$$

Every 10 mm of additional foam reduces q by ≈3.2 W/m² — significant economic benefit.

Exercises 8.2 — Thermal Systems Modelling

8.2.1 Aluminium fin: k = 205 W/m·K, d = 10 mm, L = 120 mm, h = 45 W/m²·K, T_b = 200 °C, T∞ = 30 °C. Find: (a) m (b) T at x = 60 mm (c) Q_{fin} (d) η_{fin}.

8.2.2 Copper sphere (d = 20 mm, ρ = 8 900 kg/m³, c_p = 385 J/kg·K, k = 401 W/m·K) goes from 20 °C into a furnace at 800 °C, h = 250 W/m²·K. Verify Bi < 0.1. Find: (a) τ (b) T at t = 60 s (c) dT/dt at t = 60 s.

8.2.3 Derive $\eta_{fin} = \tanh(mL)/(mL)$ for the insulated-tip fin. Show $\eta_{fin} \to 1$ as mL → 0. What does this limit represent physically?

8.2.4 Electric heater (Q_{gen} = 500 W) encased in aluminium (m = 0.8 kg, c_p = 900 J/kg·K, A = 0.06 m², h = 30 W/m²·K, T∞ = 20 °C). Find T_{ss}, τ, T(t). How long to reach 80 % of (T_{ss} − T_0)?

8.2.5 Show that total heat extracted from a cooling body (t = 0 to ∞) equals $mc_p\theta_0$. What does this represent? Verify dimensionally.

8.2.6 Composite pipe (inner steel 5 mm/k=50, insulation 50 mm/k=0.04, outer steel 3 mm/k=50) carries steam at 200 °C. Ambient 25 °C, h^{I} = 800, h_o = 10. Find U and q per unit length for inner radius 75 mm.

8.3 Fluid Dynamics Case Studies

8.3.1 The Continuity Equation — Conservation of Mass

For a fixed control volume, mass conservation (integral form) in steady, incompressible flow reduces to:

$$A_1 v_1 \;=\; A_2 v_2 \;=\; Q \;=\; \text{const} \quad [\text{m}^3/\text{s}]$$

For a duct with linearly varying diameter $D(x) = D_1 + (D_2 - D_1)x/L$:

$$\frac{dv}{dx} \;=\; -\frac{8Q(D_2 - D_1)}{\pi D(x)^3 \;\cdot\; L}$$

8.3.2 Torricelli's Theorem and Tank Draining — Full Derivation

Applying Bernoulli between the free surface and orifice exit ($A_t >> A_o$):

$$v_{\text{jet}} \;=\; C_d\sqrt{2gh} \quad \leftarrow \text{Torricelli's Theorem}$$

Mass conservation for a cylindrical tank (A_t = const):

$$A_t \;\cdot\; \frac{dh}{dt} \;=\; -C_d A_o \sqrt{2gh}$$

Separating variables and integrating:

$$h(t) \;=\; \left[\sqrt{h_0} \;-\; \frac{C_d A_o}{A_t}\sqrt{\frac{g}{2}} \;\cdot\; t\right]^2$$

$$t_{\text{empty}} \;=\; \frac{A_t}{C_d A_o}\sqrt{\frac{2h_0}{g}}$$

For a conical tank (half-angle φ, apex at bottom, $A_t(h) = \pi h^2 \tan^2\varphi$):

$$h(t) \;=\; \left[h_0{}^{5/2} \;-\; \frac{5C_d A_o\sqrt{2g}}{2\pi\,\tan^2\varphi} \;\cdot\; t\right]^{2/5}$$

8.3.3 Hagen-Poiseuille Flow — Velocity Profile Derivation

Steady, laminar, fully-developed, incompressible flow in a circular pipe of radius R. The Navier-Stokes equation in cylindrical coordinates reduces to:

$$\frac{\mu}{r} \;\cdot\; \frac{d}{dr}\left[r \;\cdot\; \frac{dv}{dr}\right] \;=\; \frac{dp}{dx} \;=\; -\frac{\Delta P}{L}$$

Integrating twice and applying symmetry + no-slip boundary conditions:

$$v(r) = \frac{\Delta P R^2}{4\mu L} \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] \quad \leftarrow \text{Parabolic profile}$$

Key derived quantities:

$$v_{\max} = \frac{\Delta P R^2}{4\mu L}; \quad v_{\text{mean}} = \frac{v_{\max}}{2}; \quad Q = \frac{\pi R^4 \Delta P}{8\mu L} \quad \leftarrow \text{Hagen – Poiseuille Law}$$

$$\tau_{\text{wall}} = \frac{\Delta P \cdot R}{2\text{L}} = \frac{4\mu v_{\text{mean}}}{R} \text{ [Pa]}$$

Friction factor f = 64/Re (Darcy-Weisbach) where Re = ρv_(2R)/μ. Laminar: Re < 2 300; turbulent: Re > 4 000.

8.3.4 Bernoulli's Equation and Flow Measurement

Steady, inviscid, incompressible flow along a streamline:

$$p + \frac{1}{2}\rho v^2 + \rho g z = \text{const}$$

Pitot tube: v = √(2Δp/ρ). Venturi meter (upstream section 1, throat 2, β = D₂/D₁):

$$Q = C_d A_2 \cdot \frac{\sqrt{2\Delta\text{p}/\rho}}{\sqrt{1 - \beta^4}} \text{ [m}^3\text{/s]}$$

Sensitivity of flow measurement:

$$\frac{dQ}{d\Delta p} = \frac{Q}{2\Delta\text{p}}$$

Accuracy is poor at low differential pressures (large dQ/dΔp).

8.3.5 Optimal Pipe Diameter — Economic Calculus

Total annual cost = capital cost (proportional to D) + pumping cost (∝ D⁻⁵):

$$C_{\text{total}}(D) = k_1 D + k_2 D^{-5}$$

Minimising with respect to D:

$$\frac{dC_{\text{total}}}{dD} = k_1 - 5k_2 D^{-6} = 0 \quad \Longrightarrow \quad D^* = \left(\frac{5k_2}{k_1}\right)^{1/6}$$

Verify minimum: d²C/dD² = 30k₂D⁻⁷ > 0 ✓. Optimal D scales with Q^1/3 (turbulent, k₂ ∝ Q²).

✦ Worked Example 8.3.1 — Draining Water Tower

Cylindrical tower: D = 8 m, h₀ = 12 m. Orifice: d_o = 150 mm, C_D = 0.62. $A_t = \pi(4)^2 = 50.27$ m²; $A_o = \pi(0.075)^2 = 0.01767$ m²

(a) Initial flow rate:

$$v_{\text{jet}} = 0.62\sqrt{2 \times 9.81 \times 12} = 9.511 \text{ m/s}; \quad Q_0 = 0.01767 \times 9.511 = 0.1680 \text{ m}^3/\text{s}$$

(b) Height versus time:

$$h(t) = [3.464 - 2.420 \times 10^{-4} \cdot t]^2 \text{ [m]}$$

(c) Time to drain from 12 m to 3 m:

$$t = \frac{A_t}{C_d A_o} \times \frac{2}{\sqrt{2g}} \times \left(\sqrt{h_0} - \sqrt{3}\right) = 3\,590 \text{ s} \approx 59.8 \text{ min}$$

(d) Total drain time:

$$t_{\text{empty}} = \frac{A_t}{C_d A_o}\sqrt{\frac{2h_0}{g}} = 7\,174 \text{ s} \approx 119.6 \text{ min}$$

(e) Flow rate at h = 3 m:

$$Q(h = 3) = 0.62 \times 0.01767 \times \sqrt{2 \times 9.81 \times 3} = 84.0 \text{ L/s}$$

Note: Q(h=3)/Q(h=12) = √(3/12) = 0.500 (Q ∝ √h, as expected).

✦ Worked Example 8.3.2 — Industrial Oil Pipeline: Laminar Flow Analysis

Oil: μ = 0.048 Pa·s, ρ = 870 kg/m³. Pipe: D = 25 mm, L = 4 m, ΔP = 12 000 Pa. R = 0.0125 m.

(a) Velocity profile:

$$v_{\max} = \frac{\Delta P R^2}{4\mu L} = \frac{12\,000 \times (0.0125)^2}{4 \times 0.048 \times 4} = 2.441 \text{ m/s}$$

$$v(r) = 2.441\left[1 - \left(\frac{r}{0.0125}\right)^2\right] \text{ m/s}$$

(b) Mean velocity and flow rate:

$$v_{\text{mean}} = \frac{2.441}{2} = 1.220 \text{ m/s}; \quad Q = \pi(0.0125)^2 \times 1.220 = 5.987 \times 10^{-4} \text{ m}^3/\text{s}$$

(c) Reynolds number (confirm laminar):

$$Re = \frac{\rho v_{\text{mean}}(2R)}{\mu} = \frac{870 \times 1.220 \times 0.025}{0.048} = 553 < 2\,300 \text{ ✓ Laminar}$$

(d) Wall shear stress:

$$\tau_{\text{wall}} = \frac{\Delta P \cdot R}{2L} = \frac{12\,000 \times 0.0125}{8} = 18.75 \text{ Pa ✓}$$

(e) Radius where v = v_ (universal result for Poiseuille flow):

$$r = \frac{R}{\sqrt{2}} = 8.84\text{ mm} \quad (\text{always } R/\sqrt{2})$$

(f) Shear rate and shear stress at r = 8 mm:

$$\frac{dv}{dr} = -\frac{2v_{\max} \cdot r}{R^2} = -250.0\text{ s}^{-1}; \quad \tau = \mu\left|\frac{dv}{dr}\right| = 0.048 \times 250 = 12.0\text{ Pa}$$

✦ Worked Example 8.3.3 — Venturi Flow Meter Calibration

Water (ρ = 1 000 kg/m³): D_1 = 100 mm, D_2 = 50 mm. Δp = 18 000 Pa, C_D = 0.98.

(a) Throat velocity:

$$v_2 = \frac{\sqrt{2\Delta p/\rho}}{\sqrt{1-\beta^4}} = \frac{\sqrt{36}}{\sqrt{0.9375}} = \frac{6.000}{0.9683} = 6.196\text{ m/s}$$

(b) Flow rate:

$$Q = C_d A_2 v_2 = 0.98 \times 1.963 \times 10^{-3} \times 6.196 = 11.93\text{ L/s}$$

(c) Sensitivity dQ/dΔp:

$$\frac{dQ}{d\Delta p} = \frac{Q}{2\Delta p} = \frac{0.01193}{2 \times 18\,000} = 3.31 \times 10^{-7}\text{ m}^3/(\text{s}\cdot\text{Pa})$$

(d) Re at throat: Re = 1 000 × 6.196 × 0.05 / 0.001 = 309 800 >> 4 000 ✓ Turbulent.

Exercises 8.3 — Fluid Dynamics

8.3.1 Conical tank (apex down, half-angle 30°, h_0 = 2 m) drains through $A_o = 5 \times 10^{-4}$ m², C_D = 0.61. Derive h(t). Find drain time. Compare to a cylinder of same volume.

8.3.2 Water (μ = 0.001 Pa·s, ρ = 1 000 kg/m³) flows in D = 50 mm pipe, L = 10 m, Q = 2 L/s. (a) Check laminar/turbulent. (b) If turbulent, estimate ΔP using Blasius: $f = 0.316Re^{-1/4}$.

8.3.3 Show that for Poiseuille flow the radius at which v = v_{-} is r = R/√2.

8.3.4 Venturi (D_1 = 80 mm, D_2 = 40 mm, C_D = 0.97) measures oil (ρ = 860 kg/m³). Δp = 25 kPa. Find Q and $v_t^{hrrr}{}_t$.

8.3.5 Show that doubling pipe radius increases Q by a factor of 16 for fixed ΔP, μ, L. How much must radius increase to double Q?

8.3.6 Sump pump: Q = 0.03 m³/s, L = 50 m horizontal pipe, PVC (f = 0.018), allowable head loss = 5 m. Find D.

8.4 Electrical Circuit Analysis (Rate-Based)

8.4.1 Constitutive Relations and KVL/KCL

Energy-storage elements in circuits are characterised by derivative-based constitutive relations:

Element	Symbol	Constitutive Relation	Energy Stored	Units
Resistor	R	v = Ri (Ohm's Law)	0 (dissipates)	Ω
Capacitor	C	i = C dv/dt	$E_C = \frac{1}{2}Cv^2$	F
Inductor	L	v = L di/dt	$E_l = \frac{1}{2}Li^2$	H

KVL: algebraic sum of voltages around any closed loop = 0. KCL: algebraic sum of currents entering any node = 0. Combining KVL/KCL with constitutive relations yields the governing ODEs; the order equals the number of independent energy-storage elements.

Time constants: RC circuit: $\tau = RC$ [s]. RL circuit: $\tau = L/R$ [s]. At $t = \tau$, the response has decayed to $e^{-1} \approx 36.8$ % of initial (or risen to 63.2 % of final).

8.4.2 RC Circuits — Charging and Discharging

Charging (source V_s, initial $v_C(0) = 0$):

$$RC \cdot \frac{dv_C}{dt} + v_C = V_s \quad \Longrightarrow \quad \tau = RC$$

$$v_C(t) = V_s\left(1 - e^{-t/\tau}\right)$$

Discharging (no source, initial $v_C(0) = V_0$):

$$v_C(t) = V_0 e^{-t/\tau}; \quad i(t) = -\frac{V_0}{R}e^{-t/\tau}$$

Energy dissipated during full discharge (counterintuitive result: independent of R):

$$E_R = \int_0^\infty i^2 R\, dt = \frac{1}{2}CV_0^{\,2} \text{ [J]}$$

All stored energy $\frac{1}{2}CV_0^2$ is dissipated in R regardless of R. Charging efficiency from a voltage source is always 50 %.

8.4.3 RL Circuits — Energisation and De-energisation

Series RL with supply V_s:

$$\frac{L}{R}\cdot\frac{di}{dt}+i=\frac{V_s}{R}\quad\Longrightarrow\quad\tau=\mathrm{L/R}$$

$$i(t)=\frac{V_s}{R}\left(1-e^{-\mathrm{t}/\tau}\right);\quad v_L(t)=V_s e^{-\mathrm{t}/\tau}$$

At t = 0 all supply voltage appears across L (inductor acts as open circuit — opposes any instantaneous change in current). At $t\to\infty$, inductor is a short circuit ($v_l\to 0$).

De-energisation through R (switch reconfigures to discharge path):

$$i(t)=I_0 e^{-\mathrm{t}/\tau};\quad v_L=-\mathrm{R}I_0 e^{-\mathrm{t}/\tau}$$

Energy stored in inductor at steady state: $E_l = \tfrac{1}{2}LI_{ss}^2$. This energy must be dissipated or diverted when the circuit is broken. Flyback diodes in relay circuits serve this protective purpose.

8.4.4 RLC Series Circuit — Second-Order Response

Natural response ($v_s = 0$):

$$\frac{d^2i^2}{dt^2}+2\alpha\frac{di}{dt}+\omega^2{}_0 i=0$$

where:

$$\alpha=\frac{R}{2\mathrm{L}}\ [\mathrm{Np/s}];\quad\omega_0=\frac{1}{\sqrt{LC}}\ [\mathrm{rad/s}];\quad\zeta=\frac{\alpha}{\omega_0}=\frac{R}{2}\sqrt{\frac{C}{L}}$$

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$, giving

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0{}^2}$$

Cases: underdamped ($\zeta < 1$, roots complex $\Longrightarrow e^{-\alpha t}$ oscillation at $\omega_D = \omega_0\sqrt{(1-\zeta^2)}$); critically damped ($\zeta = 1$, double real root $\Longrightarrow (A+Bt)e^{-\alpha t}$); overdamped ($\zeta > 1$, two distinct real negative roots).

Electrical–mechanical analogy: $m \leftrightarrow L$, $c \leftrightarrow R$, $k \leftrightarrow 1/C$, $x \leftrightarrow q$, $v \leftrightarrow i$, $F \leftrightarrow v_s$.

8.4.5 Differentiator and Integrator Op-Amp Circuits

Ideal Differentiator (input C, feedback R):

$$v_{\mathrm{out}}(t)=-\mathrm{RC}\cdot\frac{dv^{\mathrm{I}}{}_{\mathrm{n}}}{dt}$$

Gain = ωRC (increases with frequency ⟹ noise-amplifying at high f). Practical differentiator adds series R_1 to roll off gain.

Ideal Integrator (input R, feedback C):

$$v_{\text{out}}(t) = -\frac{1}{RC}\int_0^t v_{\text{in}}\,dt' + v_{\text{out}}(0)$$

Gain = 1/(ωRC) (decreases with f ⟹ low-pass). DC offset in v_{in} causes v_{out} to grow linearly (ramp). Practical integrators add large R_f parallel to C to prevent DC saturation.

✦ Worked Example 8.4.1 — Camera Flash: RC Charging Circuit

R = 2 500 Ω, C = 2 000 μF = 0.002 F, V_s = 300 V. τ = RC = 5.0 s.

(a) Response:

$$v_C(t) = 300\left(1 - e^{-t/5}\right)\text{ V}; \quad i(t) = 0.12e^{-t/5}\text{ A}$$

(b) Values at t = 2 s:

$$v_C(2) = 300(1 - e^{-0.4}) = 98.91\text{ V}; \quad i(2) = 0.12e^{-0.4} = 80.44\text{ mA}$$

(c) Time to reach 270 V:

$$e^{-t/5} = \frac{30}{300} = 0.10 \implies t = -5\ln(0.10) = 11.51\text{ s}$$

(d-f) Energy and slew rate:

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{V_s}{\tau} = 60\text{ V/s}; \quad E = \frac{1}{2}CV_s^2 = 90\text{ J}; \quad \text{Peak power} \approx 90\text{ kW}$$

Energy dissipated in R during full charging = ½CV_s^2 = 90 J (charging efficiency always 50 % for RC circuit).

✦ Worked Example 8.4.2 — Industrial Electromagnet: RL Circuit

R = 4 Ω, L = 0.5 H, V_s = 24 V. τ = L/R = 0.125 s. I_{ss} = 6 A.

(a) Current and inductor voltage:

$$i(t) = 6(1 - e^{-8t})\text{ A}; \quad v_L(t) = 24e^{-8t}\text{ V}$$

(b) Values at t = 0.2 s:

$$i(0.2) = 6(1 - e^{-1.6}) = 4.789\text{ A}; \quad v_L(0.2) = 24e^{-1.6} = 4.845\text{ V}$$

(c) Time for i = 5 A:

$$e^{-8t} = \frac{1}{6} \implies t = \frac{\ln 6}{8} = 0.2240\text{ s}$$

(d) Power at t = 0.1 s:

$$P_R = i^2R = (3.304)^2 \times 4 = 43.67\ \text{W}; \quad P_L = v_L i = 35.62\ \text{W}$$

(e) Energy stored at steady state:

$$E_L = \frac{1}{2}LI_{ss}{}^2 = \frac{1}{2} \times 0.5 \times 36 = 9\ \text{J}$$

✦ Worked Example 8.4.3 — RLC Circuit: Underdamped Natural Response

R = 40 Ω, L = 0.1 H, C = 100 μF. Initial conditions: $v_C(0) = 100$ V, $i(0) = 0$.

(a) Natural frequencies:

$$\omega_0 = \frac{1}{\sqrt{LC}} = 316.2\ \text{rad/s}; \quad \alpha = \frac{R}{2\text{L}} = 200\ \text{Np/s}; \quad \zeta = \frac{200}{316.2} = 0.6325 < 1$$

$$\Longrightarrow \text{Underdamped}$$

(b) Damped natural frequency:

$$\omega_d = \omega_0\sqrt{1-\zeta^2} = 316.2 \times 0.7746 = 244.9\ \text{rad/s}; \quad f_d = 39.0\ \text{Hz}; \quad T_d = 25.64\ \text{ms}$$

(c) Capacitor voltage (applying BCs $v_C(0) = 100$ V, $i(0) = 0$):

$$v_C(t) = e^{-200t}[100\cos(244.9t) + 81.67\sin(244.9t)]\ \text{V}$$

In amplitude–phase form:

$$v_C(t) = 129.1\, e^{-200t}\cos(244.9t - 39.26°)\ \text{V}$$

(d) First zero-crossing of v_C:

$$\cos(244.9t - 39.26°) = 0 \Longrightarrow 244.9t = 39.26° + 90° = 129.26° \Longrightarrow t = 9.22\ \text{ms}$$

(e) Energy at $t = T_D/2 = 12.82$ ms [NOTE — corrected from source: argument of cosine kept consistently in degrees]:

$$v_C(12.82\ \text{ms}) = 129.1\, e^{-2.564}\cos(180° - 39.26°)$$

$$= 129.1 \times 0.07688 \times (-\cos 39.26°) = -7.70\ \text{V}$$

$$E(T_D/2) \approx \frac{1}{2} \times 10^{-4} \times (7.70)^2$$

$$= 2.96 \times 10^{-3}\ \text{J}\ \ (99.4\ \%\ \text{of energy dissipated in half − cycle!})$$

✦ Worked Example 8.4.4 — Op-Amp Differentiator: Rate-of-Change Sensor

R = 10 kΩ, C = 0.1 μF. $RC = 1 \times 10^{-3}$ s. Input: $v_{in}(t) = 3\sin(2\,000\pi t)$ V (f = 1 000 Hz).

(a) Output voltage:

$$v_{\text{out}} = -\text{RC} \cdot \frac{dv^{\text{I}}_{\text{n}}}{dt} = -10^{-3} \times 3 \times 2\,000\pi \cos(2\,000\pi t) = -6\pi \cos(2\,000\pi t)$$
$$= -18.85 \cos(2\,000\pi t)\ \text{V}$$

(b) Gain = $\omega RC \times |V^{I}_{n}| = 2\,000\pi \times 10^{-3} \times 3 = 6\pi \approx 18.85$ V

(c) Phase: output leads input by 90° (then –ve sign adds 180° inversion).

(d) At f = 5 000 Hz: $|V_o{}^u{}_t| = 10\,000\pi \times 10^{-3} \times 3 = 30\pi \approx 94.25$ V ⟹ clipping would occur. Add $R^{I}_{n} \approx 1$ kΩ in series to limit high-frequency gain.

(e) Ramp input $v^{I}_{n} = 2t$ V: $v_o{}^u{}_t = -RC \times 2 = -2 \times 10^{-3} = -0.002$ V (DC).

Exercises 8.4 — Electrical Circuit Analysis

8.4.1 RC: R = 47 kΩ, C = 10 µF, V_s = 9 V. Find: (a) τ (b) $v_C(t)$ (c) t when v_C = 6.3 V (d) energy stored at t = τ (e) charging efficiency.

8.4.2 RL: R = 8 Ω, L = 200 mH, V_s = 12 V. Find: (a) i(t) (b) $v_l(t)$ (c) P_r at t = 50 ms (d) time for i to reach 90 % of I_{ss} (e) energy at steady state.

8.4.3 Series RLC: R = 100 Ω, L = 50 mH, C = 20 µF. Find ω_0, α, ζ. Classify response. Write general form of i(t) for natural response.

8.4.4 Derive the ODE for v_C in a parallel RLC driven by current source $i_s(t)$. Show the characteristic equation is the same as the series case (with appropriate α and ω_0).

8.4.5 Op-amp integrator: R = 100 kΩ, C = 1 µF, v^{I}_{n} = 5 V (step), $v_o{}^u{}_t(0) = 0$. Find $v_o{}^u{}_t(t)$ and the time for $|v_o{}^u{}_t|$ to reach the supply limit of ±12 V.

8.4.6 Show that energy dissipated in R when charging C from 0 to V_s through R is always $\frac{1}{2}CV_s^2$, independent of R. Interpret physically.

8.5 Structural Analysis Problems

8.5.1 Stress, Strain, and the Tangent Modulus

In a uniaxial tensile test, engineering stress σ and engineering strain ε are:

$$\boldsymbol{\sigma = F/A_0\ [Pa]}\quad \boldsymbol{\varepsilon = (L - L_0)/L_0}\ \textbf{[dimensionless]}$$

In the elastic region, Hooke's Law gives $\sigma = E\varepsilon$, with Young's modulus E [GPa]. Beyond the yield point, the tangent modulus E_t — the local slope of the σ-ε curve — is the key material parameter:

$$\boldsymbol{E_t(\varepsilon) = d\sigma/d\varepsilon}$$

In the elastic range, $E_t = E\ (constant)$. For a Ramberg-Osgood power-law material (widely used for aluminium and titanium alloys):

$$\boldsymbol{\varepsilon = \sigma/E + \left(\frac{\sigma}{K}\right)^{n}}$$

Differentiating implicitly with respect to σ:

$$d\varepsilon/d\sigma = 1/E + \left(\frac{n}{K}\right)\left(\frac{\sigma}{K}\right)^{\{n-1\}}$$

$$\boldsymbol{E_t = \left[\frac{1}{E} + \left(\frac{n}{K}\right)\left(\frac{\sigma}{K}\right)^{\{n-1\}}\right]^{\{-1\}}}$$

Note: $E_t \leq E$ always, with equality at σ = 0. As σ increases, E_t decreases — softening. This is why structures loaded beyond yield are analysed using nonlinear finite element methods.

True stress σ_{true} and true strain ε_{true} (logarithmic strain) are related to engineering values:

$$\boldsymbol{\sigma_{true} = \sigma(1 + \varepsilon)[before\ necking]\varepsilon_{true} = ln(1 + \varepsilon)}$$

$$\boldsymbol{\frac{d\sigma_{true}}{d\varepsilon_{true}} = \sigma_{true} at\ necking\ (Considère\ criterion)}$$

8.5.2 Flexure Formula and Shear Stress Distribution

For a beam in pure bending, the flexure (bending stress) formula is derived from the assumption that plane sections remain plane (Euler-Bernoulli hypothesis) and Hooke's Law:

$$\boldsymbol{\sigma(y) = My/I\ [Pa]}$$

where M is the applied bending moment, y is the distance from the neutral axis, and $I = \int y^2 dA$ is the second moment of area. The linear variation means:

$$\boldsymbol{d\sigma/dy = M/I = constant}$$

Maximum bending stress at extreme fibre y = c:

$$\boldsymbol{\sigma_{max} = \frac{Mc}{I} = \frac{M}{Z}\ where\ Z = I/c\ [m^3]\ is\ the\ elastic\ section\ modulus}$$

Shear stress distribution in a rectangular section (width b, depth d):

Applying horizontal equilibrium to a layer cut at height y from the neutral axis (Jouravski's method):

$$\tau(y) = \frac{VQ}{Ib}\ where\ Q = \int_{y}^{\frac{d}{2}} y'\, b\ \ dy' = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\boldsymbol{\tau(y) = \frac{V\left(\frac{d^2}{4} - y^2\right)}{2I}\ [parabolic\ distribution]}$$

Maximum shear stress at neutral axis (y = 0): $\tau_{max} = Vd^2/(8I) = 3V/(2A)$ (for rectangle). The shear stress is zero at the extreme fibres (y = ±d/2) — confirmed by $d\tau/dy = 0$ at $y = 0$ (maximum) and $\tau = 0\ at\ y = \pm d/2$.

For an I-section, the shear stress is constant across the flange (small contribution) and varies parabolically in the web. The abrupt change at the flange-web junction is a stress concentration that can cause horizontal shear failure in timber beams.

8.5.3 Euler Column Buckling — Derivation from the Beam-Column Equation

A slender elastic column under axial compressive load P develops lateral deflection y(x) governed by the beam-column differential equation. Taking x along the column axis and y as lateral displacement:

$$\boldsymbol{EI\ d^2y/dx^2 + P\ y = 0}$$

This is derived from the bending moment $M = EI\ y''$ and the moment equilibrium $M = -Py$ at any cross-section (for small deflections). Substituting $\lambda^2 = P/(EI)$:

$$\boldsymbol{y'' + \lambda^2\ y = 0}$$

General solution: $y(x) = A\ sin(\lambda x) + B\ cos(\lambda x)$. For pin-pin boundary conditions:

$$y(0) = 0 \rightarrow B = 0$$

$$y(L) = 0 \rightarrow A\ sin(\lambda L) = 0$$

Non-trivial solution requires $sin(\lambda L) = 0, so\ \lambda L = n\pi\ (n = 1, 2, 3, \dots)$. The critical (lowest) load corresponds to n = 1:

$\boldsymbol{P_{cr} = \pi^2 EI/L^2 \leftarrow}$ **Euler's Critical Load (pin-pin)**

The first buckled mode shape is $y(x) = A\ sin(\pi x/L)$ — a half sine wave. The second derivative of P_{cr} with respect to I: $\frac{dP_{cr}}{dI} = \frac{\pi^2 E}{L^2} > 0$ — increasing the second moment of area always increases buckling resistance. This drives the design of I-sections and hollow tubes (material placed as far from the neutral axis as possible).

Slenderness ratio $\lambda_s = \frac{L_e}{r}\ where\ r = \sqrt{(I/A)}$ is the radius of gyration and L_e is the effective length. Euler's formula is valid only when the critical stress $\sigma_{cr} = P_{cr}/A$ is below the proportional limit of the material:

$\boldsymbol{\sigma_cr = \pi^2 E/\lambda_s^2 < \sigma_P L}$

$\boldsymbol{End\ Conditions}$	$\boldsymbol{Effective\ Length\ L_e}$	$\boldsymbol{P_cr}$
$Pin - Pin$	L	$\pi^2 EI/L^2$
$Fixed - Free\ (cantilever)$	$2L$	$\pi^2 EI/(4L^2)$
$Fixed - Pin$	$\approx\ 0.699L$	$\approx\ 2.046\,\pi^2 EI/L^2$
$Fixed - Fixed$	$0.5L$	$4\pi^2 EI/L^2$

8.5.4 Combined Loading — Superposition

When a structural member is simultaneously subject to axial force P, bending moment M, and transverse shear V, the stresses are superimposed (valid in the elastic range):

$$\boldsymbol{\sigma_{total}(y)\ =\ \sigma_{axial}\ +\ \sigma_{bending}\ =\ P/A\ \pm\ My/I}$$

The critical cross-section and critical fibre must be identified by examining all load combinations. The von Mises yield criterion (for general combined stress states):

$$\boldsymbol{\sigma_V M\ =\ \sqrt{\sigma_{total}^2\ +\ 3\tau^2}} \leq \boldsymbol{\frac{f_y}{n_s}}$$

where n_s is the safety factor. Differentiating with respect to y to locate the maximum combined stress:

$$\frac{d(\sigma_V M)}{dy}\ =\frac{\sigma_{total}\ \times\frac{d\sigma_{total}}{dy}\ +\ 3\tau\ \times\frac{d\tau}{dy}}{\sigma_V M}\ =\ 0$$

Setting numerator to zero: $\sigma_{total}\ \times\ (M/I)\ =\ 3\tau\ \times\ (Vy)/(I)$ — this yields the y-location of maximum von Mises stress, which generally does not coincide with either the extreme fibre (σ_{max}) or the neutral axis (τ_{max}).

8.5.5 Deflection Optimisation — Stiffness-Weight Trade-off

For a simply supported beam of rectangular cross-section (b × d), span L, under central load P:

$$\boldsymbol{\delta_{max}\ =\frac{PL^3}{48EI}\ =\frac{PL^3}{48E\ \times\frac{bd^3}{12}}\ =\frac{PL^3}{4Ebd^3}}$$

Subject to the constraint that the cross-sectional area $A\ =\ bd\ =\ constant$ (fixed weight per unit length), we may choose the aspect ratio d/b to minimise δ:

With $b\ =\ A/d\colon \delta\ =\ PL^3/(4E(A/d)d^3)\ =\ PL^3/(4EAd^2).Since\ A\ is\ fixed, \delta\ \propto\ 1/d^2$ — minimise δ by maximising d! A deep, narrow beam is stiffer in bending than a shallow, wide beam of the same cross-sectional area.

However, a very narrow beam may fail by lateral-torsional buckling. The critical moment for lateral-torsional buckling scales as $M_{cr} \propto \frac{\sqrt{EI_y\, GJ}}{L} where\ I_y$ is the weak-axis second moment and J is the Saint-Venant torsion constant. Optimising between bending stiffness (deep beam) and lateral stability (wide beam) is a practical engineering trade-off governed by derivative analysis.

✦ Worked Example 8.5.1 — Wide-Flange Beam: Complete Stress Analysis

$W200 \times 59\ steel: I = 200 \times 10^6\ mm^4, d = 210\ mm, b_f = 205\ mm, t_f = 14.2\ mm, t_w = 9.1\ mm.$

$A = 7550\ mm^2. Loading\ at\ section: M = 180\ kN \cdot m, V = 250\ kN.$

(a) $\sigma_{max} = Mc/I = (180 \times 10^6)(105)/(200 \times 10^6) = 94.5\ MPa$

$(f_y = 355\ MPa\ for\ S355\ steel \rightarrow utilisation = 94.5/355 = 26.6\%\ in\ bending)$

(b) $\frac{d\sigma}{dy} = \frac{M}{I} = 180 \times 10^6/200 \times 10^6 = 0.90\ MPa/mm$

(bending stress changes by 0.90 MPa for every mm from neutral axis)

(c) First moment Q at neutral axis:

$$Q_{flange} = b_f \times t_f \times (\frac{d}{2} - \frac{t_f}{2}) = 205 \times 14.2 \times 97.9 = 284{,}893\ mm^3$$

$$Q_{web_{half}} = t_w \times (d/2 - t_f)^2/2 = 9.1 \times (90.8)^2/2 = 37{,}513\ mm^3$$

$$Q_N\ A = 284{,}893 + 37{,}513 = 322{,}406\ mm^3$$

$$\tau_N A = VQ/(Ib) = (250{,}000 \times 322{,}406)/(200 \times 10^6 \times 9.1) = 44.29\ MPa$$

(d) τ at flange-web junction ($y = 90.8\ mm, web\ side, b = t_w = 9.1\ mm$):

$$\tau_{junc} = VQ_f/(It_w) = (250{,}000 \times 284{,}893)/(200 \times 10^6 \times 9.1) = 39.15\ MPa$$

$$\tau\ at\ flange\ side\ (b = b_f = 205\ mm): \tau = 39.15 \times 9.1/205 = 1.74\ MPa$$

(e) $\sigma_V M$ at flange-web junction ($\sigma = My_{junc}/I = 180 \times 10^6 \times 90.8/200 \times 10^6 = 81.7\ MPa$):

$$\sigma_V M = \sqrt{(81.7^2 + 3 \times 39.15^2)} = \sqrt{(6675 + 4600)} = \sqrt{11275} = 106.2\ MPa < 355\ MPa\ \checkmark$$

✦ Worked Example 8.5.2 — SHS Column Buckling ($Pin - Pin$)

Square hollow section $150 \times 150 \times 8\,mm, S355\ steel\ (E = 200\,GPa, f_y = 355\,MPa), L = 4.5\,m.$

(a) $I = (b^4 - b_i^4)/12 = (150^4 - 134^4)/12 = (506.25 - 322.42) \times 10^6/12 = 15.32 \times 10^6\,mm^4$

$$A = 150^2 - 134^2 = 22500 - 17956 = 4544\,mm^2$$

$$r = \sqrt{(I/A)} = \sqrt{(15.32 \times 10^6/4544)} = \sqrt{3372} = 58.06\,mm$$

(b) $P_{cr} = \pi^2 EI/L^2 = \pi^2 \times 200{,}000 \times 15.32 \times 10^6/(4500^2)$

$$= 9.870 \times 3.064 \times 10^{12}/20.25 \times 10^6 = 1{,}493\,kN$$

(c) $\sigma_{cr} = P_{cr}/A = 1{,}493{,}000/4544 = 328.6\,MPa < f_y = 355\,MPa$

Elastic buckling governs ($\sigma_{cr} < f_y$) ✓

(d) Slenderness ratio $\lambda_s = \frac{L}{r} = 4500/58.06 = 77.5$

Limiting slenderness: $\lambda_{lim} = \pi\sqrt{(E/f_y)} = \pi\sqrt{(200{,}000/355)} = \pi \times 23.73 = 74.6$

Since 77.5 > 74.6, Euler formula is valid (just). ✓

(e) $dP_{cr}/dL = -2\pi^2 EI/L^3 = -2 \times 1{,}493/4500 = -0.664\,kN/mm$

Sensitivity: every 1 mm increase in column height reduces $P_{cr}\ by\ 0.667\,kN$.

A 100 mm height increase → $P_{cr}\ drops\ by\ 66.4\,kN\ (4.4\%\ reduction)$.

✦ Worked Example 8.5.3 — Crane Column: Combined Axial and Bending

I-section column: $A = 8400\,mm^2, I = 320 \times 10^6\,mm^4, c = 125\,mm.$

Loading: $P = 600\,kN\ (axial), M = 90\,kN \cdot m$ (lateral at this cross-section).

(a) $\sigma_{axial} = P/A = 600{,}000/8400 = 71.43\,MPa$ (uniform, compressive)

(b) $\sigma_{bending} = Mc/I = (90 \times 10^6)(125)/(320 \times 10^6) = 35.16\,MPa$

(±35.16 MPa on extreme fibres)

(c) Combined stresses (taking compression positive):

High-compression fibre: σ = 71.43 + 35.16 = 106.6 MPa (C) ← governs

Low-compression fibre: σ = 71.43 − 35.16 = 36.27 MPa (C)

Both fibres remain in compression — no tensile region exists.

(d) Neutral axis for combined loading (σ = 0, measuring y from NA):

$$-\frac{P}{A}+\frac{My}{I}=0\ \rightarrow\ y=(\frac{P}{A})(\frac{I}{M})=71.43\times(320/90)=71.43\times3.556=254\,mm$$

$$Since\ 254\,mm\ >\ c\ =\ 125\,mm, neutral\ axis\ is\ outside\ cross-section.\ \checkmark$$

(e) Utilisation ratio $UR = \sigma_{max}/f_y = 106.6/355 = 0.300$ (30% utilised in bending+axial)

Axial utilisation: $P/P_{cr}\ where\ P_{cr}\ =\ 1501\ kN\ (from\ 4.5\ m\ height)$: $600/1501\ =\ 0.400$

Combined check (interaction formula): $P/P_{cr}\ +\ M/M_{cr}\ \leq\ 1.0$

$$M_{cr}\ =\ f_y\ \times\ Z\ =\ 355\ \times\ (320\times10^6/125)/10^6\ =\ 355\ \times\ 2560\ =\ 908.8\,kN\cdot m$$

0.400 + 90/908.8 = 0.400 + 0.099 = 0.499 < 1.0 ✓ (Column is adequate.)

✦ Worked Example 8.5.4 — Optimal Beam Section: Stiffness Maximisation

A simply supported timber beam, $span\ L\ =\ 5\ m$, carries $P\ =\ 10\ kN$ centrally.

$E\ =\ 12\ GPa$. Cross-section is rectangular $b\times d$ with area $A\ =\ bd\ =$ 20,000 mm^2 fixed.

Deflection limit $\delta_{max}\ =\ L/300\ =\ 16.67\ mm$. Find the optimum aspect ratio d/b.

$$\delta\ =\ PL^3/(48EI)\ =\ PL^3/(4Ebd^3)\ (using\ I\ =\ bd^3/12)$$

With $b\ =\ A/d\ =\ 20{,}000/d$:

$$\delta\ =\ PL^3/(4E\ \times\ (20000/d)\ \times\ d^3)\ =\ PL^3/(4E\ \times\ 20000\ \times\ d^2)$$
$$=\ PL^3/(80000Ed^2)$$

$$Setting\ \delta\ =\ \delta_{allow}\ =\ L/300:$$

$$L/300\ =\ PL^3/(80000Ed^2)$$

$$d^2\ =\ 300PL^2/(80000E)\ =\ 300\times10{,}000\times25\times10^6/(80{,}000\times12{,}000)$$
$$=\ 75\times10^9/960\times10^6\ =\ 78.125\times10^3$$

$$d\ =\ 279.5\,mm;\ b\ =\ 20{,}000/279.5\ =\ 71.6\,mm$$

$$Aspect\ ratio\ d/b\ =\ 279.5/71.6\ =\ 3.903\ \approx\ 4$$

$$Check: I\ =\ 71.6\times(279.5)^3/12\ =\ 71.6\times21.84\times10^6/12\ =\ 130.4\times10^6\,mm^4$$

$$\delta\ =\ 10{,}000\times(5000)^3/(48\times12{,}000\times130.4\times10^6)$$
$$=\ 10^4\times1.25\times10^{11}/(48\times1.565\times10^{12})\ =\ 16.62\,mm$$
$$\approx\ 16.67\,mm\ \checkmark$$

Engineering note: Standard timber sections use $d/b \approx 3\ to\ 5$ — this analysis confirms

the practical wisdom. Wider beams ($d/b < 3$) are less stiff; deeper beams need LTB checks.

📝 Exercises 8.5 — Structural Analysis

8.5.1 $W250 \times 89$ ($I = 395 \times 10^6\ mm^4, d = 260\ mm, A = 11{,}400\ mm^2$).

$$M = 240\ kN \cdot m, V = 180\ kN. Find: (a)\sigma_{max}\ (b)\ \frac{d\sigma}{dy} \qquad (c)\tau_N\ A$$

(d) $\sigma_V M$ at the flange-web junction.

8.5.2 Hollow aluminium tube (OD = 100 mm, wall t = 8 mm, E = 70 GPa), L = 3 m, pin-pin.

Find: $(a)I$ $(b)P_{cr}$ $(c)\sigma_{cr}$ $(d)\lambda_s \frac{(e)dP_{cr}}{dI}$.

8.5.3 Derive $\tau(y) = V(d^2/4 - y^2)/(2I)$ for a rectangular section $b \times d$.

Show $\tau_{max} = 3V/(2bd)\ and\ confirm\ d(\tau)/dy = 0\ at\ y = 0$.

8.5.4 A circular steel $rod\ (d = 25\ mm, E = 200\ GPa, f_y = 250\ MPa)\ carries\ P =$ $50\ kN$ axial and M = 0.8 kN·m

Find: (a) σ_{max} (b) check yielding (c) find Euler load for L = 2 m.

8.5.5 Prove using calculus that for a fixed cross-sectional area A, the rectangular section that maximises I is the one with $d/b \to \infty\ (infinitely\ deep)$. Why is this impractical? What constraint limits d/b in practice?

8.5.6 A 6 m timber beam (b = 100 mm, d = 300 mm, $E = 12\ GPa, L = 6\ m,$ simply supported) carries 12 kN centrally. Find: (a) I (b) σ_{max} (c) δ_{max} (d) check L/300 limit

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8.6 Open-Ended Engineering Design Problems

8.6.1 Purpose and Methodology

Open-ended design problems are the hallmark of authentic engineering practice. Unlike structured textbook exercises where a unique correct answer exists and the solution path is prescribed, open-ended problems require the engineer to: (1) translate a real-world

performance requirement into a mathematical model; (2) identify and justify simplifying assumptions; (3) solve the resulting equations; (4) interpret results physically; (5) evaluate the design against criteria; and (6) propose improvements and identify limitations.

The problems below use differential calculus as the primary tool — through optimisation (dC/dx = 0), rate analysis (dT/dt, dv/ds), and sensitivity (dOutput/dParameter). Each solution framework demonstrates the complete engineering problem-solving process. A worked solution is provided for each problem, followed by extension questions.

✦ Design Problem 8.6.1 — Optimum Insulation: Cold Storage Facility

Facility: 1200 m^2 wall/roof area. Interior $T_i = -18°C$, Exterior $T_o = 38°C$ ($\Delta T = 56°C$).

Layers: $h_i = 8\ W/m^2K$ | plywood 100mm k=0.15 | foam thickness t [m] k=0.025 | $h_e =$ 20 W/m²K. COP = 2.8. Electricity: \$0.14/kWh. Insulation cost: \$85/m³. Life 25 yr, i = 6%/yr.

Thermal resistance per m²: R(t) = 1/8 + 0.1/0.15 + t/0.025 + 1/20= 0.125 + 0.667 + 40t + 0.050 = 0.842 + 40t m²K/W

Heat gain rate: Q = 1200×56/(0.842+40t) = 67200/(0.842+40t) W

Annual electricity cost (8760 hr/yr):

$$E_{annual} = Q \times 8760/(COP \times 1000 \times 0.14)^{\{-1\}} \ldots properly:$$

$$E_{annual} = [Q\ W\ /\ COP\ ] \times 8760\ hr \times 0.14\ \$/kWh \times 10^{-3}\ kW/W$$

$$= 67200/(0.842 + 40t) \times (8760 \times 0.14)/(2.8 \times 1000)$$

$$= 67200/(0.842 + 40t) \times 0.4380 = 29{,}434/(0.842 + 40t)\ \$/yr$$

$$Present\ value\ of\ energy\ (annuity\ factor\ AF = [1 - 1.06^{\{-25\}}]/0.06 = 12.783):$$

$$PV_energy = 29{,}434 \times 12.783/(0.842 + 40t) = 376{,}329/(0.842 + 40t)$$

$$Capital\ cost\ of\ foam: C_{foam} = 85 \times 1200 \times t = 102{,}000t\ \$$$

$$Total\ cost: C(t) = 102{,}000t + 376{,}329/(0.842 + 40t)$$

$$Optimise: dC/dt = 102{,}000 - 376{,}329 \times 40/(0.842 + 40t)^2 = 0$$

$$(0.842 + 40t)^2 = 376{,}329 \times 40/102{,}000 = 15{,}053{,}160/102{,}000 = 147.6$$

$$0.842 + 40t = \sqrt{147.6} = 12.15 \rightarrow 40t = 11.31 \rightarrow t^* = 0.2827\ m \approx 283\ mm$$

$$d^2C/dt^2 = 2 \times 376{,}329 \times 40^2/(0.842 + 40t^*)^3 > 0\ \checkmark(minimum)$$

$$Total\ cost\ at\ t^*: PV_energy\ =\ 376{,}329/12.15\ =\ \$30{,}972;\ C_foam\ =\ 102{,}000 \times 0.2827\ =\ \$28{,}835$$

$$C_{total}\ =\ \$59{,}807$$

Sensitivity: $dC^*\ =/dt|_{\{t=t*\pm 50mm\}}: Small\ —\ near-optimum\ zone\ is\ flat\ (insensitive).$

$$At\ t\ =\ 233\ mm\ (50mm\ less): C\ =\ 102{,}000 \times 0.233\ +\ 376{,}329/(0.842 + 9.32)\ =\ \$23{,}766 + \$37{,}290 = \$61{,}056$$

$$Only\ \$1249\ more\ —\ confirms\ wide\ near-optimum\ zone.$$

$Extension$: (1) $Repeat\ for\ electricity\ at\ \$0.25/kWh.$ (2) $Find\ the\ payback\ period\ for\ t^*\ vs\ t = 100mm.$

(3) How does optimal t change if COP drops to 2.0 (older equipment)?

✦ Design Problem 8.6.2 — Hydraulically Efficient Trapezoidal Channel

Trapezoidal channel: bottom width b, depth y, side slope 1H:1V (45°).

Cross-section: $A\ =\ by\ +\ y^2$ (given). Wetted perimeter: $P_w\ =\ b\ +\ 2\sqrt{2}\ y$.

Constraint: $A\ =\ A_0\ =\ 1.8\ m^2$ (fixed flow area). Manning's $n\ =\ 0.014, S\ =\ 0.0005$.

Goal: Maximise $Q\ =\ (1/n)A\ R^{\{2\}\{3\}} S^{\{1\}\{2\}}\ where\ R\ =\ A/P_w$.

For fixed $A\ and\ S, maximise\ R\ \leftrightarrow\ minimise\ P_w$.

Express b from area constraint: $b\ =\ (A_0/y)\ -\ y$

Substitute into P_w: $P_w(y)\ =\ A_0/y\ -\ y\ +\ 2\sqrt{2}\ y\ =\ A_0/y\ +\ (2\sqrt{2}-1)y\ =\ 1.8/y\ +\ 1.828y$

Minimise: $dP_w/dy\ =\ -1.8/y^2\ +\ 1.828\ =\ 0$

$$y^2\ =\ 1.8/1.828\ =\ 0.9847\ \rightarrow\ y^*\ =\ 0.9923\ m$$

$$d^2P_w/dy^2\ =\ 3.6/y^3\ >\ 0\ \checkmark(minimum)$$

b* = 1.8/0.9923 − 0.9923 = 1.8140 − 0.9923 = 0.8217 m

Check: A = 0.8217×0.9923 + (0.9923)² = 0.8151 + 0.9847 = 1.800 m² ✓

$$P_w^*\ =\ 1.8/0.9923\ +\ 1.828 \times 0.9923\ =\ 1.814\ +\ 1.814\ =\ 3.628\ m$$

$$R^* = A/P_w^* = 1.800/3.628 = 0.4963\,m$$

$$Q^* = \left(\frac{1}{0.014}\right)(1.800)(0.4963)^{\{2\}\{3\}}(0.0005)^{\{1\}\{2\}}$$

$$= 71.43 \times 1.800 \times 0.6226 \times 0.02236 = 1.788\,m^3/s$$

Compare with suboptimal y = 0.8 m (fixed arbitrarily):

$$b = \frac{1.8}{0.8} - 0.8 = 1.45\,m;\ P_w = \frac{1.8}{0.8} + 1.828 \times 0.8 = 2.25 + 1.462 = 3.712\,m$$

$$R = \frac{1.80}{3.712} = 0.4850;\ Q = 71.43 \times 1.800 \times 0.4850\text{^}\{0.667\} \times 0.02236 = 1.748\,m^3/s$$

Penalty for non-optimal design: (1.788–1.748)/1.788 = 2.2% flow reduction.

Extension: (1) Derive the result for a rectangle (side slopes = 0): show b = 2y.

(2) Show the optimal trapezoid is inscribed in a semicircle of radius y.

(3) Find the optimal rectangular channel for the same A0 and compare Q.

✦ Design Problem 8.6.3 — PID Controller: Derivative Term Analysis

Process: first-order thermal system with $\tau_p = 120\,s, K_p = 2°C/W$.

Step response: $T(t) = T_{set}\left(1 - e^{\left\{-\frac{t}{\tau_p}\right\}}\right)$ for unit step in heater power.

PID control law: $u(t) = K_c[e + (1/T_I)\int e\,dt + T_D\,de/dt]$

where $e = T_{set} - T\,(error).\,Design: K_c = 2\,W/°C, T_D = 30\,s.$

$$At\ t = 60\,s\ after\ a\ 100°C\ setpoint\ step\ from\ T_0 = 0°C:$$

$$T(60) = 100\left(1 - e^{\left\{-\frac{60}{120}\right\}}\right) = 100\left(1 - e^{\{-0.5\}}\right) = 100 \times 0.3935 = 39.35°C$$

$$e(60) = 100 - 39.35 = 60.65°C$$

Rate of temperature rise: $\frac{dT}{dt} = (K_p/\tau_p) \times e^{\left\{-\frac{t}{\tau_p}\right\}} \times (heater\ power)$ …

More directly: in closed loop with proportional action only:

$\frac{dT}{dt}|_{\{t=60\}} \approx (T_{set} - T(60))/\tau_p \times K_p = 60.65 \times 2/120 = 1.011°C/s$ (open loop estimate)

$\frac{de}{dt} = -dT/dt = -1.011°C/s$ (error decreasing)

PID output components at t = 60 s:

$$u_P = K_c \times e = 2 \times 60.65 = 121.3\,W$$

$$u_D = K_c \times T_D \times de/dt = 2 \times 30 \times (-1.011) = -60.66\,W$$

$$u_{total}\,(P+D) = 121.3 - 60.7 = 60.6\,W\ (derivative\ term\ halves\ the\ control\ action!)$$

Physical interpretation:

Without D: full 121 W applied despite T rising rapidly → likely overshoot

With D: derivative 'sees' that T is rising fast and pre-emptively reduces heater power

Result: smoother approach to setpoint, reduced overshoot, less oscillation.

Noise sensitivity: differentiator amplifies noise with $gain = \omega T_D$

At f = 10 Hz (ω = 62.8 rad/s): gain = 62.8×30 = 1884× (extreme noise amplification)

Practical solution: filtered derivative $u_D = K_c\,T_D\,s/(1 + T_D\,s/N)\ with\ N = 10$

Maximum gain of filtered $D = K_c \times N = 2 \times 10 = 20\,W/°C\ (bounded)$

Extension: (1) Calculate the noise RMS amplification for $N = 5\ vs\ N = 20$.

(2) Design a setpoint filter to avoid derivative kick on setpoint changes.

✦ Design Problem 8.6.4 — Minimum-Weight Fatigue-Constrained Tension Link

Fluctuating load: F(t) = 60,000 + 20,000 sin(ωt) N

Steel: $f_u = 900\,MPa, f_y = 750\,MPa, S_e = 360\,MPa.$

Safety factors: $n_f = 2.0\ (fatigue), n_y = 1.5\ (yielding).$

Stresses: $\sigma_{mean} = 60{,}000/A;\ \sigma_a = 20{,}000/A$

Goodman fatigue criterion: $\sigma_a/S_e + \sigma_{mean}/f_u = 1/n_f$

$$20{,}000/(360A) + 60{,}000/(900A) = 1/2$$

$$(55.56 + 66.67)/A = 0.500$$

$$A_{fat} = 122.22/0.500 = 244.4\,mm^2$$

$$Yield\ criterion: (\sigma_{mean} + \sigma_a)/f_y = 1/n_y$$

$$A_{yld} = n_y(F_{mean} + F_a)/f_y = 1.5 \times 80{,}000/750 = 160.0\,mm^2$$

Governing: A_{min} = max(244.4, 160.0) = 244.4 mm² (fatigue governs)

Minimum diameter: d = √(4A/π) = √(4×244.4/π) = √311.0 = 17.64 mm → use 18 mm

Sensitivity analysis — derivatives of required area:

$$dA/d(F_a) \;=\; n_f/S_e \;=\; 2.0/360 \;=\; 5.556\times10^{-3}\; mm^2/N$$

$$dA/d(F_{mean}) \;=\; n_f/f_u \;=\; 2.0/900 \;=\; 2.222\times10^{-3}\; mm^2/N$$

Ratio = 5.556/2.222 = 2.5 — alternating load is 2.5× more damaging than mean load.

This is a direct consequence of the Goodman line gradient.

Modified Goodman vs Gerber: Gerber uses $(\sigma_{mean}/f_u)^2$ instead of linear term.

Gerber $A_f at$: $\sigma_a/S_e \;+\; (\sigma_{mean}/f_u)^2 \;=\; 1/n_f$

20,000/(360A) + (60,000/(900A))² = 0.5 — nonlinear in A, solve numerically.

(Goodman is more conservative — standard design choice.)

Extension: (1) Find A_{min} if F_a increases to 35,000 N. (2) What S_e is needed for A = 200 mm²?

(3) How does the result change if f_u increases to 1200 MPa (higher-strength steel)?

✦ Design Problem 8.6.5 — Solar Collector Tilt Optimisation (Nairobi, 1.3°S)

Flat-plate collector at latitude $\varphi \;=\; 1.3°S.\, G_H \;=\; 850\; W/m^2$ (beam irradiance at noon).

For south-facing collector tilted at angle β from horizontal, solar declination δ:

Angle of incidence $\theta_i \;=\; |\varphi \;-\; \beta \;+\; \delta|$ (simplified, noon condition, φ in degrees).

Irradiance on tilted surface: $G_T \;=\; G_H\, cos(\theta_i)$.

Optimise for equinox (δ = 0):

$$G_T(\beta) \;=\; 850\; cos|1.3° \;-\; \beta|$$

$$dG_T/d\beta \;=\; 850\; sin(1.3° \;-\; \beta) \;\times\; sign(1.3° - \beta) \;=\; 0$$

$$\rightarrow\; \beta^* \;=\; 1.3°\; (collector\; tilted\; at\; latitude\; angle \;—\; equatorial\; rule)$$

$$G_T^* \;=\; 850\; cos(0) \;=\; 850\; W/m^2$$

$$\frac{d^2G_T}{d\beta^2} \;=\; -850\; cos(0) \;=\; -850 \;<\; 0\; \checkmark(maximum)$$

Seasonal variation:

June solstice $\delta = +23.45°$: $\theta_i = |1.3 - \beta + 23.45| = |24.75 - \beta|$; $\beta^* = 24.75°$

Dec solstice $\delta = -23.45°$: $\theta_i = |1.3 - \beta - 23.45| = |\beta + 22.15|$; $\beta^* = -22.15°$ (not practical)

Annual average: optimum $\beta^* \approx \varphi = 1.3°$

Sensitivity at equinox: $G_T(\beta) = 850\ cos(\beta - 1.3°)$

For β = 0 (horizontal): $G_T = 850\ cos(1.3°) = 850 \times 0.99974 = 849.8\ W/m^2$

Loss relative to $\beta^* = 1.3°$: $(850.0 - 849.8)/850 = 0.024\%$ — negligible!

$dG_T/d\beta|_{\{\beta=0\}} = -850\ sin(1.3°) = -850 \times 0.02269 = -19.3\ W/m^2$ per degree

$At\ \beta = 10°: loss = 850(1 - cos(8.7°)) = 850 \times 0.01147 = 9.75\ W/m^2$ $(1.15\%\ loss)$

Conclusion: Nairobi's near-equatorial location means tilt angle is almost irrelevant for annual irradiance — a horizontal mount loses < 2% annually vs optimal tilt.

Economic recommendation: use horizontal mounting (simplest, lowest cost, no self-cleaning issues).

Extension: (1) Find optimal monthly β for Jan, Apr, Jul, Oct in Nairobi.

(2) Calculate annual energy using $E = \int_0^{8760} G_T\,(t)\ dt\ for\ \beta = 0\ vs\ \beta = 1.3°$.

(3) Extend to include diffuse radiation $G_d = 150\ W/m^2$ (isotropic model).

📝 Open-Ended Design Exercises 8.6

8.6.1 THERMAL: An LED module (48 W) needs an aluminium heat sink ($k = 205\ W/m \cdot K$, $h = 35\ W/m^2 \cdot K$). 12 rectangular fins, $t = 2\ mm, w = 25\ mm, pitch\ p = 8\ mm$.

Base must stay < 75°C (ambient 25°C). Find minimum fin height H.

Then minimise total heat sink mass by choosing H and the number of fins N.

8.6.2 KINEMATICS: A race car has thrust $F(v) = 12{,}000 - 80v^2\ N, m = 750\ kg$.

(a) $\frac{dv}{dt} = F(v)/m$ — $find\ v_T$ (b) $solve\ v(s)\ using\ v\ dv/ds = F(v)/m$

(c) $distance\ to\ reach$ 80% $of\ v_T$ $(d)\ jerk\ at\ t = 0$ $(e)\ critique\ model$.

8.6.3 ELECTRICAL: Design a Butterworth RC low-pass filter, $f_c = 3\ kHz$.

$v_{out}/v_{in} = 1/\sqrt{1+\left(\frac{f}{f_c}\right)^2}$. (a) Find −3 dB cutoff formula

(b) select R, C (c) find $\frac{d(attenuation_{dB})}{df}$ beyond f_c

(d) for 40 dB/decade roll-off, how many identical stages are needed?

8.6.4 STRUCTURAL: A traffic sign post ($hollow\ steel\ tube, E = 200\ GPa$) carries a 1.2 m × 0.8 m sign at 5 m height. Design wind pressure = 1.2 kPa.

Deflection limit H/150. Find minimum I, select tube section, check buckling under axial weight ($sign + post = 350\ N$). Comment on governing failure mode.

8.6.5 FLUIDS: A pump delivers Q = 0.04 m³/s of water through a branching network:

Branch A: L = 400 m, required $Q_A = 0.025\ m^3/s$.

Branch B: L = 600 m, required $Q_B = 0.015\ m^3/s$.

Both branches have the same allowable head loss = 6 m.

$$Use\ \Delta P = \frac{8fL\rho Q^2}{\pi^2 D^5}\ with\ f = 0.020.\ Find\ D_A\ and\ D_B.$$

Apply $\frac{dD}{dQ}$ to assess sizing sensitivity.

Chapter 8 Summary and Key Results

This chapter has demonstrated that differential calculus is the unifying mathematical language of quantitative engineering. Across six distinct domains, the derivative appears in structurally analogous — though physically diverse — governing equations. The complete picture:

Section	***Physical Principle***	***Governing Equation***	***Key Result***
8.1 *Kinematics*	*Newton's 2nd Law*	$m\,\dot{v} = F(v,s,t)$	$v_T = \sqrt{(mg/k)};\ v(t) = v_T\ tanh(\frac{gt}{v_T})$
8.2 *Lumped Thermal*	*1st Law (closed system)*	$mc_p\,\theta = -hA\theta$	$T(t) = T_\infty + \theta_0\,(e^{\{-\frac{t}{\tau}\}});\ \tau = \frac{mc_p}{hA}$
8.2 *Fin Equation*	*Steady energy balance*	$\frac{d^2\theta}{dx^2} = m^2\theta$	$\frac{\theta}{\theta_b} = \frac{cosh[m(L-x)]}{cosh(mL)}$

Section	Physical Principle	Governing Equation	Key Result
8.3 *Draining Tank*	*Mass conservation*	$A_t\,\dot{h} = -C_d\,A_o\sqrt{(2gh)}$	$h(t) = [\sqrt{h_0} - (C_dA_o/A_t)\sqrt{(g/2)}t]^2$
8.3 *Pipe Flow*	*Navier − Stokes (laminar)*	$\mu\nabla^2 v = \nabla p$	$Q = \pi R^4 \Delta P/(8\mu L)$
8.4 *RC Circuit*	$KVL + i = C\,dv/dt$	$\tau\,dv_C/dt + v_C = V_s$	$v_C = V_s(1 - e^{\{-\frac{t}{\tau}\}});\ \tau = RC$
8.4 *RLC Circuit*	*KVL (2nd order)*	$\ddot{y} + 2\alpha\dot{y} + {\omega_0}^2 y = 0$	$\zeta = \alpha/\omega_0;\ \omega_d = \omega_0\sqrt{(1-\zeta^2)}$
8.5 *Beam Bending*	*Flexure formula*	$\sigma = \frac{My}{I};\ \tau = \frac{VQ}{Ib}$	$\frac{d\sigma}{dy} = \frac{M}{I};\ \tau_max = \frac{3V}{2A}$
8.5 *Column Buckling*	*Euler BVP*	$EI\,y'' + Py = 0$	$P_cr = \pi^2 EI/L^2$
8.6 *Optimisation*	*Minimise cost/weight*	$\frac{dC}{dD} = 0$	$D^* = \left(\frac{5k_2}{k_1}\right)^{\{\frac{1}{6}\}};\ t^*_{ins} = analytical$

The six-step modelling methodology — (1) identify conservation principle, (2) apply to differential element, (3) take limit to obtain governing equation, (4) solve analytically, (5) interpret physically, (6) optimise — is the foundation of engineering science. Mastering this process, rather than memorising formulas, is the primary goal of this chapter.

The electrical-mechanical analogy (mass ↔ inductance, damping ↔ resistance, stiffness ↔ 1/capacitance) and the thermal-electrical analogy (temperature ↔ voltage, heat flow ↔ current, thermal resistance ↔ electrical resistance) demonstrate the deep mathematical unity underlying apparently different physical phenomena. Engineers who recognise these analogies can transfer solution techniques across domains — one of the most powerful skills in applied mathematics.

Chapter Review Problems

📑 Integrated Review Problems — Chapter 8

R7.1 A parachutist ($m = 90\,kg, k = 0.35\,kg/m$) opens the chute at $v_0 = 50\,m/s$.

(a) Find v_T (b) Set up and solve the ODE for v(t) with initial condition $v_0 \neq 0$

[Hint: use partial fractions differently from the zero-IC case] (c) Find t to reach

within 2% $of\ v_T$ (d) Compute jerk at t = 0 and at t = 10 s.

R7.2 An aluminium fin ($k = 200\,W/m\cdot K, 15 \times 2\,mm, L = 60\,mm, h = 60\,W/m^2\cdot K$).

$T_b = 150°C, T_\infty = 20°C. Find: (a) m \frac{(b)\theta(x)}{\theta_b} (c) Q_f in (d) \eta_f.$

R7.3 A tank ($A_t = 4\ m^2, h_0 = 5\ m$) drains through $A_o = 0.005\ m^2$ ($C_d = 0.62$) while water is pumped in at $Q_{in} = 0.01\ m^3/s$. Derive the ODE for h(t).

Find the equilibrium depth h_{eq}. Is the equilibrium stable? (Check d(ḣ)/dh.)

R7.4 Ideal LC circuit: $L = 80\ mH, C = 50\ \mu F. Initial: v_C(0) = 50\ V, i(0) = 0$. Find,

$(a)\omega^0 \quad (b) v_C(t)\ and\ i(t) \quad (c)\ i_{max} \quad (d)\ show\ total\ energy = ½Cv_0^2\ always.$

$(e)\ sketch\ the\ phase\ portrait\ i\ vs\ v_C$..

R7.5 Crane column $W310 \times 129$ ($A = 16{,}500\ mm^2, I = 308 \times 10^6\ mm^4, c = 165\ mm$).

$P = 800\ kN, M = 120\ kN \cdot m, L_e = 6\ m, E = 200\ GPa$..

Find: $(a)\sigma_{axial} \quad (b)\sigma_{bending\ at\ extreme\ fibres} \quad (c)\sigma_{max\ combined}$

$(d) P_{cr} \quad (e) interaction\ ratio \frac{P}{P_{cr}} + \frac{M}{M_{cr}} \leq 1.0$..

R7.6 (Design) An injection moulding nozzle ($m = 0.12\ kg, c_p = 480\ J/kg \cdot K$, $A_{conv} = 0.025\ m^2, h = 18\ W/m^2 \cdot K$) *has heater power* $\dot{Q}_{gen} = 450\ W$,
Also loses heat

by conduction $\dot{Q}_{cond} = k_{cond}\, A_{cond} \frac{\Delta T}{L} = 16 \times 8 \times 10^{-6} \times \frac{\Delta T}{0.1}. T_\infty = 22°C,$.

$target\ T_{ss} = 230°C$..

(a) Set up energy ODE with two loss terms (b) find T_{ss} (c) find τ

(d) is 230°C achievable? If not, what $\dot{Q}_{gen}$ is needed?

R7.7 Derive from first principles the Hagen-Poiseuille equation $Q = \frac{\pi R^4 \Delta P}{8\mu L}$.

Show each integration step and the boundary conditions applied.

Verify dimensional homogeneity of the final result.

R7.8 For a series RLC circuit (R = 200 Ω, L = 100 mH, C = 10 μF), find:

(a) ω_0, α, ζ (b) ω_d if underdamped (c) $v_C(t)\ for\ v_C(0) = 200V, i(0) = 0$

(d) time and value of first peak (e) energy dissipated in first cycle.

Selected Answers

Review R7.1 — Parachutist with non-zero initial velocity

With v₀ ≠ 0 and quadratic drag, the ODE m dv/dt = mg − kv² separates as before, but integration constants differ. Using partial fractions:

$$\int dv/(v_T^2 - v^2) = (1/2v_T)[ln|(v_T + v)/(v_T - v)|] = gt/v_T^2 + C$$

$$At\ t = 0: \left(\frac{1}{2v_T}\right) ln\left|\frac{v_T + v_0}{v_T - v_0}\right| = C$$

$$\boldsymbol{v(t) = v_T \cdot \frac{\left[v_T\, tanh\left(\frac{gt}{v_T} + atanh\left(\frac{v_0}{v_T}\right)\right)\right]}{v_T} = v_T\, tanh\left(\frac{gt}{v_T} + \varphi_0\right)}$$

where $\varphi^0 = atanh\left(\frac{v^0}{v_T}\right) = atanh\left(\frac{50}{v_T}\right). With\ v_T = \sqrt{90 \times \frac{9.81}{0.35}} = \sqrt{2523} = 50.23\frac{m}{s}$: $\varphi_0 = atanh(50/50.23) = atanh(0.9954) = 3.027\ rad$

$v(t) = 50.23\ tanh(9.81t/50.23 + 3.027) t\ to\ within\ 2\%\ of\ v_T: tanh(0.1953t + 3.027) = 0.98$

$0.1953t + 3.027 = atanh(0.98) = 2.298\ldots \rightarrow\ t\ is\ negative \rightarrow\ already\ past\ v_T\ from\ the\ start!\ Since\ v_0 = 50\ m/s \approx v_T = 50.23\ m/s$, parachutist is almost immediately at terminal velocity

Review R7.4 — Ideal LC Circuit

$$\omega_0 = 1/\sqrt{(LC)} = 1/\sqrt{(0.08 \times 50 \times 10^{-6})} = 1/\sqrt{(4 \times 10^{-6})} = 500\ rad/s$$

$$v_C(t) = 50\ cos(500t) V\ i(t) = -C\frac{dv_C}{dt} = 50 \times 10^{-6} \times 500 \times 50\ sin(500t) = 1.25\ sin(500t)\ A$$

$$i_{max} = \frac{v_C(0)}{\sqrt{\frac{L}{C}}} = \frac{50}{\sqrt{\frac{0.08}{50 \times 10^{-6}}}} = \frac{50}{\sqrt{1600}} = \frac{50}{40} = 1.25\ A\ \checkmark$$

$E_{total} = ½Cv_C^2 + ½Li^2 = ½ \times 5 \times 10^{-5} \times 2500cos^2 + ½ \times 0.08 \times 1.5625sin^2(500t)$

$= 0.0625cos^2(500t) + 0.0625sin^2(500t)\ldots = 0.0625\ J\ always\ (const)\ \checkmark [using\ ½Cv_0{}^2 = ½ \times 10^{-4} \times 2500 = 0.125J]$

Review R7.7 — Hagen-Poiseuille Derivation Checklist

Starting from the axisymmetric Navier-Stokes equation (fully developed, steady, incompressible):

$$0 = -\frac{\partial p}{\partial x} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right] \rightarrow \frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \left(\frac{\frac{1}{\mu}dp}{dx} = -\frac{\Delta P}{\mu L}\right.$$

$$Step\ 1: Multiply\ by\ r: \frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\left(\frac{\Delta P}{\mu L}\right)r$$

$$Step\ 2: Integrate: r\frac{dv}{dr} = -\left(\frac{\Delta P}{2\mu L}\right)r^2 + C^1;\ at\ r = 0: \frac{dv}{dr} = 0 \rightarrow C_1 = 0$$

$$Step\ 3: \frac{dv}{dr} = -\left(\frac{\Delta P}{2\mu L}\right)r$$

$$Step\ 4: Integrate: v = -\left(\frac{\Delta P}{4\mu L}\right)r^2 + C_2;\ at\ r = R: v = 0 \rightarrow C_2 = \Delta P R^2/(4\mu L)$$

$$Step\ 5: v(r) = \frac{\Delta P R^2}{4\mu L} \times \left[1 - \left(\frac{r}{R}\right)^2\right];\ v_{max} = \frac{\Delta P R^2}{4\mu L}$$

$$Step\ 6: Q = \int_o^R v(r) \times 2\pi r\, dr = 2\pi\frac{\Delta P R^2}{4\mu L}\int_0^R \left(1 - \frac{r^2}{R^2}\right) r\, dr$$

$$= 2\pi\frac{\Delta P R^2}{4\mu L} \times \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 2\pi\frac{\Delta P R^2}{4\mu L} \times \frac{R^2}{4} = \frac{\pi R^4 \Delta P}{8\mu L}\ \checkmark$$

$$Dimensional\ check: [Q] = \frac{[m^4][Pa]}{[Pa\cdot s][m]} = m^4 \cdot \frac{\frac{N}{m^2}}{N\cdot\frac{s}{m^2}\cdot m} = \frac{m^4}{s\cdot m} \times \frac{m}{m} = m^3/s\ \checkmark$$

Appendix A — Master Derivative Reference Table

This table collects all differentiation rules derived in Chapters 2–4. All standard results are listed in their most general chain-rule form (where u = u(x) is a differentiable function of x).

A.1 Basic Rules

Rule	Formula	Notes / Conditions
Constant	$d/dx\,[c] = 0$	*c any constant*
Power	$d/dx\,[x^n] = n\,x^{n-1}$	$n \in \mathbb{R}$
Scalar Multiple	$d/dx\,[c \cdot f] = c \cdot f'$	*c constant*
Sum	$d/dx\,[f + g] = f' + g'$	*Linearity*
Difference	$d/dx\,[f - g] = f' - g'$	*Linearity*
Product	$d/dx\,[fg] = f'g + fg'$	*Both differentiable*
Quotient	$d/dx\,[f/g] = (f'g - fg') / g^2$	$g \neq 0$
Chain	$d/dx\,[f(u)] = f'(u) \cdot u'$	$u = u(x)$

A.2 Exponential and Logarithmic Functions

Rule	Formula	Notes
Natural exponential	$d/dx\,[e^x] = e^x$	*Self − referential*
General exponential	$d/dx\,[a^x] = a^x \ln a$	$a > 0, a \neq 1$
Chain — exp	$d/dx\,[e^u] = e^u \cdot u'$	$u = u(x)$
Natural log	$d/dx\,[\ln x] = 1/x$	$x > 0$
General log	$d/dx\,[\log_a x] = 1/(x \ln a)$	$a > 0, a \neq 1$
Chain — ln	$d/dx\,[\ln u] = u'/u$	$u > 0$
Variable exponent	$d/dx\,[u^v] = u^v[v' \ln u + v \cdot u'/u]$	*Logarithmic diff.*

A.3 Trigonometric Functions

Rule	Formula	Notes
Sine	$d/dx\,[\sin x] = \cos x$	*x in radians*
Cosine	$d/dx\,[\cos x] = -\sin x$	*x in radians*
Tangent	$d/dx\,[\tan x] = \sec^2 x$	$x \neq \pi/2 + n\pi$
Cosecant	$d/dx\,[\csc x] = -\csc x \cot x$	$x \neq n\pi$
Secant	$d/dx\,[\sec x] = \sec x \tan x$	$x \neq \pi/2 + n\pi$
Cotangent	$d/dx\,[\cot x] = -\csc^2 x$	$x \neq n\pi$
Chain form	$d/dx\,[\sin u] = \cos u \cdot u'$	*All 6 — replace x with u*

A.4 Inverse Trigonometric Functions

Rule	Formula	Domain
$arcsin$	$d/dx\,[arcsin\,x] = 1/\sqrt{(1-x^2)}$	$\lvert x\rvert < 1$
$arccos$	$d/dx\,[arccos\,x] = -1/\sqrt{(1-x^2)}$	$\lvert x\rvert < 1$
$arctan$	$d/dx\,[arctan\,x] = 1/(1+x^2)$	$All\ x$
$arccsc$	$d/dx\,[arccsc\,x] = -1/(\lvert x\rvert\sqrt{(x^2-1)})$	$\lvert x\rvert > 1$
$arcsec$	$d/dx\,[arcsec\,x] = 1/(\lvert x\rvert\sqrt{(x^2-1)})$	$\lvert x\rvert > 1$
$arccot$	$d/dx\,[arccot\,x] = -1/(1+x^2)$	$All\ x$

A.5 Hyperbolic Functions

Rule	Formula	Definition
$sinh$	$d/dx\,[sinh\,x] = cosh\,x$	$sinh\,x = (e^x - e^{-x})/2$
$cosh$	$d/dx\,[cosh\,x] = sinh\,x$	$cosh\,x = (e^x + e^{-x})/2$
$tanh$	$d/dx\,[tanh\,x] = sech^2\,x$	$tanh\,x = sinh\,x/cosh\,x$
$csch$	$d/dx\,[csch\,x] = -csch\,x\,coth\,x$	$csch\,x = 1/sinh\,x$
$sech$	$d/dx\,[sech\,x] = -sech\,x\,tanh\,x$	$sech\,x = 1/cosh\,x$
$coth$	$d/dx\,[coth\,x] = -csch^2\,x$	$coth\,x = cosh\,x/sinh\,x$

A.6 Higher-Order and Special Rules

Rule	Formula	Conditions
Leibniz rule	$(fg)^{(n)} = \Sigma\, C(n,k)\, f^{(k)}\, g^{(n-k)}$	$k = 0$ *to* n
Implicit diff.	$dy/dx = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$	$F(x,y) = 0$
Parametric	$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$	$dx/dt \neq 0$
Parametric 2nd	$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$	$dx/dt \neq 0$
Log differentiation	$y = f^g \rightarrow \ln y = g \ln f \rightarrow \dfrac{y'}{y}$	*Variable exponent*
Partial $\partial f/\partial x$	*Hold all vars const. except* x	*Multivariable*
Gradient	$\nabla f = \left(\frac{\partial f}{\partial x}\right) i + \left(\frac{\partial f}{\partial y}\right) j + \left(\frac{\partial f}{\partial z}\right) k$	*Direction of max increase*

Appendix B — Notation and Symbols Guide

The following table lists all major notation systems and symbols used in this book. Multiple notations for the same concept are equivalent; the choice is governed by context and convention.

B.1 Derivative Notation Systems

Notation	Symbol	Usage context
Leibniz	$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^ny}{dx^n}$	Engineering analysis; makes variable of differentiation explicit. Preferred in dif
Lagrange	$f'(x), f''(x), f$	Pure mathematics and function − centred arguments. Compact for repeated derivatives.
Newton (d	$\dot{y}, \ddot{y}$	Time derivatives in dynamics and control. Convention: one dot $= \frac{d}{dt}$, two dots $= \frac{d^2}{dt^2}$.
Euler (D)	Df, D^2f, D^nf	Operator notation; common in differential equation theory.
Partial	$\frac{\partial f}{\partial x}, f_x, D_xf$	Multivariable functions; holds all other variables constant.

B.2 Greek Letters Used in Engineering Calculus

Letter	Name	Symbol	Common Engineering Use

$\boldsymbol{\alpha}$	$Alpha$	$\alpha \ / \ A$	$Thermal\ diffusivity, damπng\ coefficient, angle$
$\boldsymbol{\beta}$	$Beta$	$\beta \ / \ B$	$Thermal\ expansion\ coefficient, frequency\ ratio$
$\boldsymbol{\gamma}$	$Gamma$	$\gamma \ / \ \Gamma$	$Shear\ strain, ratio\ of\ specific\ heats\ (Cp/Cv)$
$\boldsymbol{\delta}$	Δ	$\delta \ / \ \Delta$	$Small\ increment, Dirac\ \delta\ function, deflection$
$\boldsymbol{\varepsilon}$	E	ε	$Normal\ strain, small\ positive\ number\ (\varepsilon - \delta\ proofs)$
$\boldsymbol{\zeta}$	$Zeta$	ζ	$Damπng\ ratio\ in\ vibration\ theory$
$\boldsymbol{\eta}$	Eta	η	$Efficiency, dynamic\ viscosity\ (alternative)$
$\boldsymbol{\theta}$	Θ	θ	$Angle, temperature\ excess\ above\ ambient$
$\boldsymbol{\lambda}$	$Lambda$	λ	$Eigenvalue, wavelength, decay\ constant$
$\boldsymbol{\mu}$	Mu	μ	$Dynamic\ viscosity, friction\ coefficient, micro - prefix$
$\boldsymbol{\nu}$	Nu	ν	$Poisson's\ ratio, kinematic\ viscosity$
$\boldsymbol{\xi}$	Xi	ξ	$Generalised\ coordinate, dummy\ integration\ variable$
$\boldsymbol{\pi}$	Π	π	$Ratio\ of\ circumference\ to\ diameter\ \approx\ 3.14159\ldots$
$\boldsymbol{\rho}$	Rho	ρ	$Density\ (mass\ per\ unit\ volume)$
$\boldsymbol{\sigma}$	$Sigma$	$\sigma \ / \ \Sigma$	$Normal\ stress, standard\ deviation, summation$
$\boldsymbol{\tau}$	T	τ	$Shear\ stress, time\ constant, torque$
$\boldsymbol{\varphi}$	Phi	$\varphi \ / \ \Phi$	$Phase\ angle, electric\ potential, heat\ flux$
$\boldsymbol{\omega}$	Ω	$\omega \ / \ \Omega$	$Angular\ frequency\ (rad/s), electrical\ resistance\ (\Omega)$

Appendix C — Further Reading and References

The following references are recommended for readers who wish to deepen their understanding of the toπcs covered in this book. They range from classical texts to modern engineering mathematics references.

Calculus Foundations

Stewart, J. *Calculus: Early Transcendentals (9th ed.)* Cengage Learning, 2020. *Comprehensive single and multivariable calculus. Excellent worked examples and problem sets.*

Apostol, T. M. *Calculus, Vol. 1 & 2* Wiley, 1967. *Rigorous and thorough. Standard reference for the mathematically inclined engineer.*

Sπvak, M. *Calculus (4th ed.)* Publish or Perish, 2008. *The gold standard of rigorous single-variable calculus. Challenging but deeply rewarding.*

Engineering Mathematics

Kreyszig, E. *Advanced Engineering Mathematics (10th ed.)* Wiley, 2011. *The definitive reference for engineering mathematics at undergraduate and graduate level.*

Stroud, K. A. & Booth, D. J. *Engineering Mathematics (8th ed.)* Macmillan, 2020. *Highly accessible, programmed-learning format. Ideal for self-study.*

Riley, K. F., Hobson, M. P. & Bence, S. J. *Mathematical Methods for Physics and Engineering (3rd ed.)* Cambridge University Press, 2006. *Comprehensive coverage of all mathematical methods encountered in physics and engineering.*

Multivariable Calculus and Vector Analysis

Marsden, J. E. & Tromba, A. *Vector Calculus (6th ed.)* W. H. Freeman, 2011. *Standard multivariable calculus text with strong geometric insight.*

Schey, H. M. *Div, Grad, Curl and All That (4th ed.)* Norton, 2005. *Intuitive introduction to vector calculus for physicists and engineers.*

About the Author

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Saina Philiph Kipkosgei is a scholar in mathematics education, engineering mathematics, optimisation modelling, and educational technology. He is affiliated with the University

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This book reflects his commitment to making advanced mathematical concepts accessible, applicable, and intellectually rigorous for engineering students, educators, and researchers.

Concluding Remarks

Differential calculus is far more than a branch of mathematics; it is one of the principal intellectual tools through which modern civilisation understands motion, change, optimisation, and design. From the stability of bridges and aircraft to thermal systems, electronic circuits, fluid transport, and artificial intelligence models, the derivative remains central to engineering analysis and scientific discovery.

The purpose of this text has been not merely to present differentiation techniques, but to cultivate mathematical reasoning grounded in physical interpretation and engineering application. Students who master these foundations are well prepared to progress into integral calculus, differential equations, numerical analysis, optimisation theory, and advanced engineering mathematics.

Mathematics rewards persistence. Every theorem understood, every derivation reconstructed, and every problem solved independently strengthens analytical capacity and professional competence. It is hoped that this book contributes meaningfully to that journey.

— End of Book —